Static Output Feedback Stabilization of Linear Systems with Multiple Delays

Danilo Braghini[1], Eduardo S. Tognetti[2] and Matthew M. Peet[3]

***Abstract*— This work proposes a new procedure for the stabilization of time-delay systems using Static Output Feedback (SOF) control. A previous convex optimization approach to SOF for Ordinary Differential Equations (ODEs) is extended to time-delay systems through the use of a proposed state-space representation. This approach is based on solving two convex optimization problems, which are extensions of Linear Matrix Inequalities (LMIs) to infinite-dimensional systems. The first problem is stabilization under state feedback control; the second problem takes advantage of the Projection Lemma, which is extended here from matrices to Partial Integral (PI) operators. Finally, the results are compared with other SOF solutions for systems with delay found in the literature, showing significant reduction in conservatism.**

I. Introduction

Systems with time delays are common in control engineering problems since signals in natural phenomena are not transmitted instantaneously but rather with some latency. Moreover, measurements, data acquisition, and processing of these signals often introduce delays. Examples of models where delays cannot be neglected include milling and cutting [1], and ventilation in mining rooms [2], where the delays can destabilize and compromise the performance of the controlled system. While state feedback control requires either full real-time state information or the use of a state estimator, Static Output Feedback (SOF) relies entirely on available measurements. Furthermore, previous results suggest that delays can make systems stabilizable under SOF [3], [4], motivating the search for SOF controllers for time-delay systems. However, despite the simplicity and reduced implementation costs, the SOF control problem is known to be non-convex and thus computationally hard to solve even for linear ordinary differential equations (ODEs) [5].

The difficulty in solving the SOF control problem is evident by the limitations in existing approaches. For example, the computational package presented in [4] performs analysis and control synthesis for systems with multiple delays in the state and output, and can be used to compute SOF controllers. Even though this method is quite general, it relies on solving a non-convex problem, which may lead to suboptimal local solutions, limiting the effectiveness of the approach.

On the other hand, convex optimization in the form of Linear Matrix Inequality (LMIs) is a well-established paradigm in control theory, as LMIs can be efficiently solved by interior-point methods [6]. Thus, many attempts of convexifying the SOF problem have been proposed, at the expense of introducing conservatism. Unfortunately however, the existing LMIs for SOF control of systems with delays are conservative and only valid for restricted classes of single-delay systems. For example, in [7], a sliding-mode-based SOF controller was proposed and a robust solution is presented in terms of LMIs for uncertain systems with time-varying delays. Nevertheless, the synthesis condition is derived using conservative inequalities to bound the derivative of the Lyapunov function and is valid for the case of a single state delay. On the other hand, in [8], a different approach based on LMIs is proposed but it is only valid for positive systems with a single delay in the output.

For linear time-invariant ODEs, a synthesis approach for stabilizing SOF control based on LMIs was presented in [9] using the Projection Lemma [10]. The key idea is to require stability under full state feedback and SOF using a common quadratic Lyapunov function. Combined with the Projection Lemma, this approach allows one to compute the state feedback and the SOF controllers sequentially, solving an LMI in each step. The simplification of using a single quadratic Lyapunov function in both control problems resembles the widely-used notion of quadratic stability [11]. Inspired by the two-stage solution of [9], a condition to compute an SOF controller for systems with delays was presented in [12]. However, the synthesis conditions are based on a well-known delay-independent LMI, and, in most cases, an iterative algorithm is necessary to choose an optimal first-stage controller for the second step. Moreover, the condition presented in [12] is only valid for systems with a single state delay.

Fortunately, convex optimization conditions for control of a general class of time-delay systems can be derived using the Partial Integral Equation (PIE) representation [13]. The algebraic structure of the Partial Integral (PI) operators that parametrize PIEs allows the reformulation of control problems – with little conservatism and without discretization of the distributed state [14] – as convex optimization problems with a finite number of variables and constraints called Linear Partial Integral Inequalities (LPIs). In this sense, LPIs are an extension of LMIs from matrices to PI operators. PIEs can be used to represent systems with multiple discrete and distributed delays, neutral-type systems, and even PDEs with delays [15], [16]. As long as the system admits a PIE repre-

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[1]Danilo Braghini{dbraghini@asu.edu} and [3] Matthew M. Peet{mpeet@asu.edu} are with the School for Engineering of Matter, Transport and Energy at Arizona State University, Tempe, AZ, USA.

[2]Eduardo S. Tognetti{estognetti@ene.unb.br} is with the Electrical Engineering Department, at University of Brasília, Brasília, DF, Brazil.

sentation, full-state feedback control synthesis for PDEs and systems with delay can be reduced to existing LPIs [17], [18]. However, the existing state feedback controllers computed with PIEs framework are hard to implement since the full-state may not be available and the implementation of PIE estimators in digital hardware requires real-time numerical integration of an infinite-dimensional auxiliary system [19].

The goal of the paper is to propose a procedure to SOF control of systems with multiple state and output delays, inspired by the work of [9] and previous developments in the PIEs framework. The resulting contributions are: 1) The extension of the Projection Lemma from the matrix algebra to the PI algebra; 2) A two-step procedure less conservative than previous results to compute a stabilizing SOF controller valid for a wide class of systems with delays, where each step involves solving a convex optimization problem. The new SOF solution presented herein takes full advantage of the algebraic parametrization provided by PIEs by requiring no discretization schemes in the synthesis and implementation.

First, the PIE representation provides an algebraic parametrization akin to the state-space representation of ODEs. Then, the Projection Lemma is extended from matrices to the PI algebra. Due to the Projection Lemma, a bilinear inequality is obtained where the bilinearity can be circumvented by leveraging a previous result: an LPI from [18] is first solved for the stabilizing full-state feedback control, and this controller is then used as an input to the main optimization problem, which reduces to a new LPI. Finally, numerical examples from the literature validate the proposed solution.

Notation: $L_2^m[a,b]$ is the space of Lesbegue square-integrable $\mathbb{R}^m$-valued functions on spatial domain $s \in [a,b]$, endowed with the standard inner product. $W^{mK}[a,b]$ is the Sobolev space of continuously differentiable Lesbegue square-integrable $\mathbb{R}^{mK}$–valued functions on $s \in [a,b]$. For simplicity, we hereafter denote the space $\mathbb{R}^m \times L_2^n[a,b]$ as $\mathbb{R}L_2^{m,n}[a,b]$ and $\mathbb{R}^m \times W^{mK}[a,b]$ as $\mathbb{R}W^{m,mk}[a,b]$; both the spatial domain and dimensions may be omitted when clear from context. Note that $\mathbb{R}L_2$ is a Hilbert space when endowed with the usual inner-product. The same is valid for $\mathbb{R}W$. For Hilbert spaces X, Y, $\mathcal{L}(X,Y)$ denotes the set of bounded linear operators from X to Y with $\mathcal{L}(X) := \mathcal{L}(X,X)$. We use the calligraphic font (e.g. $\mathcal{A}$) to represent such bounded linear operators. For any $\mathcal{A} \in \mathcal{L}(Y,X)$, $\mathcal{A}^*$ denotes the adjoint operator and for self adjoint operators, $\mathcal{A} \succcurlyeq 0$ means $\langle \mathbf{x}, \mathcal{A}\mathbf{x} \rangle \geq 0$ for all $\mathbf{x} \in X$.

II. Problem Formulation

Consider the delay differential equation:

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{K} A_i x(t-\tau_i) + Bu(t), t \geq 0 \tag{1}$$

$$y(t) = Cx(t) + \sum_{i=1}^{K} C_i x(t-\tau_i)$$

$$x(t) = x_0(t), \quad -\tau_K \leq t \leq 0$$

where $x(t) \in \mathbb{R}^m$, $u(t) \in \mathbb{R}^{n_u}$ is the control input, $y(t) \in \mathbb{R}^{n_y}$ is the measured output, $x_0 \in W^m[-\tau_K, 0]$ is a given function, and, for convenience, $\tau_1 < \tau_2 < \cdots < \tau_K$. A, A_i, B, C, and C_i are matrices of appropriate dimensions.

In the SOF problem, we want to find a matrix $L \in \mathbb{R}^{n_u \times n_y}$ for the control law $u(t) = Ly(t)$ such that the following closed-loop system is exponentially stable.

$$\dot{x}(t) = (A + BLC)\, x(t) + \sum_{i=1}^{K} (A_i + BLC_i)\, x(t-\tau_i), \tag{2}$$

$$x(t) = x_0(t), \quad -\tau_K \leq t \leq 0.$$

The first challenge is the infinite-dimensional nature of the linear system (2), which does not allow the usual matrix parametrization of the state-space in time-domain. An algebraic parametrization is possible by the PIE state space representation presented in Sec. III, which allows one to extend LMI methods to infinite-dimensional systems. Nevertheless, even in the case of systems without delay, $A_i = C_i = 0$, the problem is known to be intractable. Specifically, from the Lyapunov theorem, the closed-loop linear system is exponentially stable if and only if there exists a positive definite matrix $P \succ 0$ such that $(A + BLC)^T P + P\,(A + BLC) \prec 0$. The key observation is that, in contrast to the state feedback case, the position of L in the matrices of the delay system in Eq. (2) does not allow a simple variable substitution trick to linearize the inequality, as done in the state feedback control problem [20].

In [9], the authors propose a condition that unifies the state and output feedback problems in one matrix inequality for the ODE case. Given a matrix K, the state feedback control law $u(t) = Kx(t)$ produces an exponentially stable closed-loop system if and only if there exists a positive definite matrix $P_2 \succ 0$ such that $(A + BK)^T P_2 + P_2\,(A + BK) \prec 0$. Now, let us consider $P_2 = P$. Then, we have to verify the following matrix inequalities

$$(A + BLC)^T P + P\,(A + BLC) \prec 0, \tag{3}$$
$$(A + BK)^T P + P\,(A + BK) \prec 0$$

As shown in Sec. IV, the Projection Lemma allows us to rewrite two bilinear inequalities like (3) as a single bilinear inequality with an additional variable. The main advantage of the new form is not extending the condition to one single inequality, but the rearrangement of the variables. Note that the bilinearity in eq (3) is due to the Lyapunov variable P and the SOF control gain L in the term $PBLC$. On the other hand, on the new extended bilinear inequality L is removed from the bilinear terms and replaced by K. Consequently, by computing the state feedback gain K in a first step and using the result as an input, the new inequality becomes an LMI and the SOF gain L can be computed. Thus, the synthesis problem can be reduced to solving two LMIs sequentially.

The results presented in Sec. IV generalizes a sufficient version of the Projection Lemma from matrices to PI operators and use the lemma to derive a condition for systems with multiple delays in the state and output, preserving the infinite-dimensional nature of the delay. The low conservativeness of the solution is demonstrated by comparison with existing non-convex and convex solutions in Sec. V.

III. PIEs, Representation of DDEs, and Stability Conditions

In this section, the PIE representation is briefly introduced, which allows the formulation of an LPI, a generalization of an LMI, to solve the SOF problem for the delay system in Eq. (1). The main result presented in Sec. IV also requires us to define exponential stability and recall two prior results: an LPI for stability analysis and an LPI for full-state feedback stabilization, which are reproduced here for completeness.

A. 4-PI Operators

In the PIE representation presented in this section, the delay systems are parametrized by the class of partial integral operators defined in the following. For a more comprehensive and general overview, refer to Sec. II of [21].

Definition 1: Given a matrix P and polynomials Q_1, Q_2, R_0, R_1, and R_2, a 4-PI operator $\mathcal{P} = \mathit{\Pi}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\}\end{array}\right] \subset \mathcal{L}(\mathbb{R}L_2^{m,n}, \mathbb{R}L_2^{p,q})$ is such that

$$\left(\mathcal{P}\begin{bmatrix} x \\ \mathbf{x}\end{bmatrix}\right)(s) := \begin{bmatrix} Px + \int_{-1}^{0} Q_1(\theta)\mathbf{x}(\theta)d\theta \\ Q_2(s)x + \mathcal{R}\mathbf{x}(s)\end{bmatrix}, \text{where}$$

$$\left(\mathcal{R}\mathbf{x}\right)(s) = R_0(s)\mathbf{x}(s) + \int_{-1}^{s} R_1(s,\theta)\mathbf{x}(\theta)d\theta + \int_{s}^{0} R_2(s,\theta)\mathbf{x}(\theta)d\theta.$$

Furthermore, the set of 4-PI operators with dimensions m, n, q, p is denoted $\mathbf{\Pi}_{q,n}^{p,m}$.

If $p = m$ and $q = n$, this set of 4-PI operators is closed under composition, addition, and adjoint; explicit formulae for these operations can be obtained in terms of the polynomial matrices used to parameterize them [21]. Concatenation and inversion of PI operators are also defined in some cases; the reader may find precise definitions and formulae in [21] and [18], respectively. The associated dimensions (m, n, p, q) are inherited from the dimensions of the constant matrix $P \in \mathbb{R}^{p\times m}$ and polynomial matrices $Q_1(s) \in \mathbb{R}^{p\times n}$, $Q_2(s) \in \mathbb{R}^{q\times m}$, and $R_0(s), R_1(s,\theta), R_2(s,\theta) \in \mathbb{R}^{q\times n}$.

In the case where a dimension is zero, we use $\emptyset$ in place of the associated parameter with zero dimension. For example, the particular case of $n = q = 0$ makes

$$P = \mathit{\Pi}\left[\begin{array}{c|c} P & \emptyset \\ \hline \emptyset & \{\emptyset\}\end{array}\right] : \mathbb{R}^m \to \mathbb{R}^p.$$

Thus, any matrix can clearly be associated with a 4-PI operator.

B. Delay Systems Formulation

Next, we need to show how 4-PI operators can be used to represent systems with multiple delays. First, we show that for any $\{A, B, C, A_i, C_i\}$, the delay system in Eq. (1) can be represented as a PDE coupled with an ODE as follows:

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{K} A_i\phi_i(t,-1) + Bu(t) \qquad (4)$$
$$y(t) = Cx(t) + \sum_{i=1}^{K} C_i\phi_i(t,-1)$$
$$\partial_t\phi_i(s,t) = \frac{1}{\tau_i}\partial_s\phi_i(s,t), \quad -1 \le s < 0, \quad t \ge 0$$
$$\phi_i(0,t) = x(t),$$

for all $i = 1, \cdots, K$, where $\phi_i(s,t) = x(t + s\tau_i)$. We define the PDE state $\mathbf{v}(t) := (x(t), \phi_1(\cdot,t), \ldots, \phi_K(\cdot,t)) \in \mathbb{R}X$ in the Hilbert space $\mathbb{R}X = \{\mathbf{v} \in \mathbb{R}W^{m,mK}[-1,0] : \phi_i(0,t) = x(t)\}$.

Lemma 2: For any $x_0 \in W^m[-\tau_K, 0]$, $u \in L_2[0,\infty)$, if x, y satisfy the delay system in Eq. (1) under input u and initial condition x_0, then $\mathbf{v}(\cdot,t) = (v_1(t), \mathbf{v}_2(\cdot,t))$ and y satisfy the PDE in Eq. (4) under input u and initial conditions $\mathbf{v}_2(s,0) = (x(s\tau_1), \ldots, x(s\tau_K))$ for $s \in [-1,0)$, and $v_1(0) = x(0)$. Conversely, if $\mathbf{v} = (v_1, \eta_1, \ldots, \eta_K)$, y satisfy the PDE in Eq. (4), under input u and initial conditions $\eta_i(s,0) \in W^m[-1,0]$, $1 \le i \le K$, and $v_1(0) \in \mathbb{R}^m$, then $x(t) = \eta_i(0,t)$ for all $t \ge 0$, and y satisfy the delay system in Eq. (1) under initial condition $x_0(t) = \eta_K(t/\tau_K, 0)$, for all $t \in [-\tau_K, 0]$.

Proof: The proof follows from the application of Lemmas 1 and 3 from [15] in the particular case where the delay channel of the differential-difference equation is $r_i(t) = x(t)$, with $i = 1, \cdots, K$. ■

Our next goal is to show that the delay system in Eq. (1) admits an equivalent PIE representation of the form

$$\partial_t(\mathcal{T}\mathbf{x}(t)) = \mathcal{A}\mathbf{x}(t) + \mathcal{B}u(t),$$
$$y(t) = \mathcal{C}\mathbf{x}(t), \qquad (5)$$

where $\mathcal{T}, \mathcal{A}, \mathcal{B}, \mathcal{C}$ are operators as in Def. 1, $\mathbf{x}(t) \in \mathbb{R}L_2^{m,mK}[-1,0]$. With the result from Lem. 2, we just need to show the equivalence between the PDE representation and the PIE. Specifically, for any $\{A, B, C, A_i, C_i\}$, the delay system in Eq. (1) admits a PIE representation given by Fig. (7), and the solutions of the DDE and the PIE are equivalent as follows.

Lemma 3: Suppose System (5) is defined by $\{\mathcal{T}, \mathcal{A}, \mathcal{B}, \mathcal{C}\}$ given by Fig. (7). For any $\mathbf{x}_0 \in \mathbb{R}L_2^{m,mK}[-1,0]$ and $u \in L_2[0,\infty)$, if x, y satisfies System (1) under input u and initial condition x_0, then $\mathbf{x}$ and y satisfy System (5), for $\mathbf{x}(t) = (x(t), \dot{x}(t + s\tau_1), \ldots, \dot{x}(t + s\tau_K))$, $t \ge 0$, and $-1 \le s \le 0$, under input u and initial conditions $x(0)$ and $\dot{x}(s\tau_i)$, for $1 \le i \le K$. Likewise, if $\mathbf{x}(t) = (x(t), \dot{x}(t + s\tau_1), \ldots, \dot{x}(t+s\tau_K))$ and y satisfy System (5), under input u and initial condition $\mathbf{x}(0) = (x(0), \dot{x}(s\tau_1), \ldots, \dot{x}(s\tau_K))$, then $x(t)$ satisfy System (1) under input u and initial condition $x_0(t) = x(t/\tau_K)$, for all $t \in [-\tau_k, 0]$.

Proof: This result may be proved by first applying Lem. 2 of this paper and then using Lemma 4 of [15]. For completeness, we reproduce here the result from [15] in the particular case of the delay differential Eq. (1).

After applying Lem. 2, it remains to show the equivalence between the PDE and the PIE representations. First, observe that $\phi_i(t,-1)$ can be written in terms of 4-PI operators; from the Fundamental Theorem of Calculus (FTC), we have $\phi_i(t,-1) = \phi_i(t,0) + \int_0^{-1}\partial_\theta\phi_i(t,\theta)d\theta$, where $\phi_i(t,0) = x(t)$. Substituting into the dynamic equations of Eq. (4) with the definition of $\mathbf{v}(t)$, defining $\mathbf{x}(t) = (x(t), \partial_s\phi_1(\cdot,t), \ldots, \partial_s\phi_K(\cdot,t))$, and using the addition and composition formulas for 4-PI operators presented in [21], we have

$$\partial_t(\mathbf{v}(t)) = \underbrace{\mathit{\Pi}\left[\begin{array}{c|c} A+\sum_{i=1}^K A_i & -\left[A_1 \cdots A_K\right] \\ \hline 0 & \{I_\tau, 0, 0\}\end{array}\right]}_{\mathcal{A}}$$

$$+ \underbrace{\mathit{\Pi}\left[\begin{array}{c|c} B & \emptyset \\ \hline 0 & \{\emptyset\}\end{array}\right]}_{\mathcal{B}} u(t). \tag{6}$$

Similarly, in the output equation,

$$y(t) = \underbrace{\mathit{\Pi}\left[\begin{array}{c|c} C+\sum_{i=1}^K C_i & -\left[C_1 \cdots C_K\right] \\ \hline \emptyset & \{\emptyset\}\end{array}\right]}_{\mathcal{C}} \mathbf{x}(t).$$

The final step is to eliminate $\mathbf{v}(t)$ from Eq. (6). For this aim, note that $\mathbf{v}(t) = \mathcal{T}\mathbf{x}(t)$ where $\mathcal{T}$ is a 4-PI operator obtained by another application of the FTC: $\phi_i(t,s) = x(t) - \int_s^0 \partial_\theta \phi_i(t,\theta) d\theta$. Then,

$$\mathbf{v}(t) = \underbrace{\mathit{\Pi}\left[\begin{array}{c|c} I_m & 0 \\ \hline \left[I_m \quad \cdots \quad I_m\right]^T & \{0, 0, -I_{mK}\}\end{array}\right]}_{\mathcal{T}} \mathbf{x}(t).$$

■

Remark 1: Even though both $\mathbf{x}(t)$ and $\mathbf{x}(t)$ are elements of an infinite-dimensional Hilbert space, $\mathbb{R}X$ has additional continuity and boundary conditions imposed by the derivative operators, whereas $\mathbb{R}L_2$ is an analogous infinite-dimensional space to the state-space $\mathbb{R}^m$. This key fact allows us to remove continuity and boundary constraints from the optimization problems derived using PIEs framework.

Naturally, closed-loop systems can also be represented as PIEs. Given $\mathcal{K} \in \mathbf{\Pi}_{0,mK}^{n_u,m}$, the full-state feedback control law $u(t) = \mathcal{K}\mathbf{x}(t)$ produces the closed-loop system

$$\partial_t \mathcal{T}\mathbf{x}(t) = (\mathcal{A} + \mathcal{B}\mathcal{K})\, \mathbf{x}(t), \tag{8}$$

whereas the static output feedback $u(t) = Ly(t)$ yields the closed-loop delay system in Eq. (2), with the PIE representation

$$\partial_t \mathcal{T}\mathbf{x}(t) = (\mathcal{A} + \mathcal{B}L\mathcal{C})\, \mathbf{x}(t). \tag{9}$$

C. An LPI for Exponential Stability

After establishing the equivalence between multiple delay systems and PIEs, we can derive convex optimization conditions to prove the stability of the open-loop systems. The control problem solved in Sec. IV is to find a matrix L such that the closed-loop PIE Eq. (9) achieves exponential stability as defined in the following.

Definition 4 (Exponential Stability of a delay system): We say that the delay system in Eq. (1) with $u = y = 0$ is exponentially stable with decay rate $\alpha > 0$ if there is a constant M such that for any initial condition $x_0 \in \mathbb{R}W$, $x(t)$ satisfies $\|x(t)\|_{\mathbb{R}^m} \le M \|x_0\|_{\mathbb{R}W} e^{-\alpha t}$ for all $t \ge 0$.

Definition 5 (Exponential Stability of a PIE): We say that the PIE Eq. (5) is exponentially stable with decay rate $\alpha > 0$ if there is a constant M such that for any initial state $\mathcal{T}\mathbf{x}(0)$, with $u = y = 0$, the solution $\mathbf{x}(t)$ satisfies $\|\mathcal{T}\mathbf{x}(t)\|_{\mathbb{R}L_2} \le M \|\mathbf{x}(0)\|_{\mathbb{R}L_2} e^{-\alpha t}$, for all $t \ge 0$.

Next, Lem. 6 provides a key observation: we show that stability of the PIE is sufficient for stability of the delay system. In fact, the results presented in this paper are derived in terms of PIE stability. Therefore, the results remain valid for any system with a PIE representation rather than restricted to the delay system in Eq. (1).

Lemma 6: If the PIE Eq. (5), with $\{\mathcal{A}, \mathcal{T}\}$ given by Fig. (7) is exponentially stable, then the delay system in Eq. (1) is exponentially stable.

Proof: Assume that Eq. (5) is exponentially stable as in Def. 5 and recall that the PDE state $\mathbf{v}(t) = \mathcal{T}\mathbf{x}(t)$. It follows trivially from the norm definitions that $\|x(t)\|_{\mathbb{R}^m} \le \|\mathcal{T}\mathbf{x}(t)\|_{\mathbb{R}L_2}$ for all $t \ge 0$. On the other hand, from [21], $\|\mathbf{x}(0)\|_{\mathbb{R}L_2} = \|\mathcal{T}\mathbf{x}(0)\|_{\mathbb{R}X}$, implying

$$\|x(t)\|_{\mathbb{R}^m} \le M \|\mathcal{T}\mathbf{x}(0)\|_{\mathbb{R}X} e^{-\alpha t},$$

for any $\|\mathcal{T}\mathbf{x}(0)\|_{\mathbb{R}X}$. But, Lemma 17 of [21] imply that $\|\mathcal{T}\mathbf{x}(0)\|_{\mathbb{R}X} \le \|\mathcal{T}\mathbf{x}(0)\|_{\mathbb{R}W}$. Then,

$$\|x(t)\|_{\mathbb{R}^m} \le M \|\mathbf{v}(0)\|_{\mathbb{R}W} e^{-\alpha t},$$

where $\|\mathbf{v}(0)\|_{\mathbb{R}W} = \|x(0)\|_{\mathbb{R}^m} + \|\phi(0,\cdot)\|_{\mathbb{R}W}$. From direct application of Lem. 2, $\mathbf{v}(0) = x_0$, completing the proof. ■

The first LPI we present was derived in [18]. For completeness, we reproduce the result here and rely on Lem. (6) to clarify that the condition can be used to test stability of delay systems as in Def. 4.

Lemma 7: Let $\mathcal{A}, \mathcal{T}$ be 4-PI operators of appropriate dimensions. If there exist constants $\delta, \alpha > 0$ and a self-adjoint 4-PI operator $\mathcal{P} = \mathcal{P}^*$ such that $\mathcal{P} \succcurlyeq \delta I$ and

$$\mathcal{A}^*\mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\mathcal{A} \preccurlyeq -2\alpha \mathcal{T}^*\mathcal{P}\mathcal{T}, \tag{10}$$

then the PIE Eq. (5) defined by $\{\mathcal{A}, \mathcal{T}\}$ is exponentially stable with decay rate α.

Proof: The proof can be found in [18]. ■

The LPI in Lem. 7 is used to prove the SOF synthesis condition presented in the main result of this work.

D. An LPI for State Feedback Stabilization

To address the SOF problem, we first determine a full-state feedback gain to serve as input data for the SOF synthesis condition proposed in Sec. IV. To solve the full-state feedback problem, we adopt an approach similar to that used for deriving the corresponding LMI for ODEs. The key idea of the following result is to use the Dual PIE of the closed-loop PIE Eq. (8), as defined in [18]:

$$\partial_t(\mathcal{T}^*\bar{\mathbf{x}}(t)) = (\mathcal{A} + \mathcal{B}\mathcal{K})^*\, \bar{\mathbf{x}}(t). \tag{11}$$

Lemma 8: Consider 4-PI operators $\mathcal{A}, \mathcal{B}, \mathcal{T}$ of appropriate dimensions. If there exists a 4-PI operator $\mathcal{Z}$, a self-adjoint 4-PI operator $\mathcal{P} = \mathcal{P}^*$ and constants $\delta, \alpha > 0$ such that $\mathcal{P} \succcurlyeq \delta I$ and

$$\mathcal{A}\mathcal{P}\mathcal{T}^* + \mathcal{T}\mathcal{P}\mathcal{A}^* + \mathcal{B}\mathcal{Z}\mathcal{T}^* + \mathcal{T}\mathcal{Z}^*\mathcal{B}^* \preccurlyeq -2\alpha \mathcal{T}\mathcal{P}\mathcal{T}^*, \tag{12}$$

then the PIE System (8) defined by $\{\mathcal{A}, \mathcal{B}, \mathcal{T}, \mathcal{K}\}$, where $\mathcal{K} = \mathcal{Z}\mathcal{P}^{-1}$, is exponentially stable with decay rate α

Proof:

The proof can be found in [18]. ■

The LPI in Lem. 8 was proved in [18] and can be used to compute a stabilizable full-state feedback controller to the

$$\mathcal{T} = \mathit{\Pi}\left[\begin{array}{c|c} I_m & 0 \\ \hline \begin{bmatrix} I_m & \cdots & I_m \end{bmatrix}^T & \{0,0,-I_{mK}\} \end{array}\right], \mathcal{A} = \mathit{\Pi}\left[\begin{array}{c|c} A+\sum_{i=1}^K Ai & -\begin{bmatrix} A_1 \cdots A_K \end{bmatrix} \\ \hline 0 & \{I_\tau, 0, 0\} \end{array}\right], \mathcal{B} = \mathit{\Pi}\left[\begin{array}{c|c} B & \emptyset \\ \hline 0 & \{\emptyset\} \end{array}\right], \quad (7)$$

$$\mathcal{C} = \mathit{\Pi}\left[\begin{array}{c|c} C+\sum_{i=1}^K Ci & -\begin{bmatrix} C_1 \cdots C_K \end{bmatrix} \\ \hline \emptyset & \{\emptyset\} \end{array}\right], I_\tau = \begin{bmatrix} \frac{1}{\tau_1} I_m & & \\ & \ddots & \\ & & \frac{1}{\tau_K} I_m \end{bmatrix},$$

Fig. 1: Conversion formulae from PDE to PIE.

PIE system in Eq. (8), which is the first-step to solve the procedure for SOF presented in Sec. IV.

IV. Main Results

After introducing a state-space representation of delay systems, showing the equivalence of exponential stability between the representations and recalling an LPI for stability, we are ready to present the main results of this work. First, we partially extend the Projection Lemma from matrices to 4-PI operators, proving only sufficiency. Then, we use this lemma to derive a convex optimization solution for SOF of time-delay systems in two steps.

A. Projection Lemma for PIs

First, recall the Projection Lemma of [11].

Lemma 9: [11] Given a symmetric matrix Q and two matrices U and V of column dimension m; there exists an unstructured matrix F that satisfies

$$Q + U^T F^T V + V^T F U \prec 0,$$

if and only if the following projection inequalities with respect to F are satisfied

$$N_u^T Q N_u \prec 0, \quad N_v^T Q N_v \prec 0,$$

where N_u and N_v are arbitrary matrices whose columns form a basis of the null spaces of U and V, respectively.

Next, we propose a sufficient condition that extends Lem. 9, widely applied for analysis and control of ODEs, for PIEs. However, we need to first define a right annihilator of a 4-PI operator.

Definition 10: Given $\mathcal{R} \in \mathbf{\Pi}_{q,n}^{p,m} \subset \mathcal{L}(\mathbb{R}L_2^{m,n}, \mathbb{R}L_2^{p,q})$, we say $\mathcal{S} \in \mathbf{\Pi}_{n,l}^{m,k} \subset \mathcal{L}(\mathbb{R}L_2^{k,l}, \mathbb{R}L_2^{m,n})$ is a right annihilator of $\mathcal{R}$ if $\mathcal{R}(\mathcal{S}\mathbf{x}) = 0 \in \mathbb{R}L_2^{p,q}$, $\forall \mathbf{x} \in \mathbb{R}L_2^{k,l}$ and $\mathcal{R}^*\mathcal{R} \succcurlyeq \epsilon I$ for some positive constant $\epsilon > 0$.

Then, sufficiency of the Projection Lemma, Lem. 9, can be extended to the algebra of 4-PI operators as follows.

Lemma 11: Consider 4-PI operators $\mathcal{V} \in \mathbf{\Pi}_{q,n}^{p,m}$, $\mathcal{U} \in \mathbf{\Pi}_{s,n}^{r,m}$, and a self-adjoint 4-PI operator $\mathcal{Q} = \mathcal{Q}^* \in \mathbf{\Pi}_{n,n}^{m,m}$. Let $\mathcal{R}$ and $\mathcal{S}$ be right annihilators of $\mathcal{U}$ and $\mathcal{V}$, respectively. Then, if there exists a 4-PI operator $\mathcal{X}$, of appropriate dimensions, such that

$$\mathcal{Q} + \mathcal{U}^*\mathcal{X}\mathcal{V} + \mathcal{V}^*\mathcal{X}^*\mathcal{U} \preccurlyeq 0, \quad (13)$$

then the two LPIs hold:

$$(\mathcal{S})^*\mathcal{Q}\mathcal{S} \preccurlyeq 0, \quad (\mathcal{R})^*\mathcal{Q}\mathcal{R} \preccurlyeq 0. \quad (14)$$

Proof: By definition, Eq. (13) implies

$$\langle \mathbf{x}, (\mathcal{Q} + \mathcal{U}^*\mathcal{X}\mathcal{V} + \mathcal{V}^*\mathcal{X}^*\mathcal{U})\mathbf{x}\rangle \leq 0,$$

for all $\mathbf{x} \in \mathbb{R}L_2^{m,n}$. Now, note that, in particular, when $\mathbf{x} = \mathcal{S}\mathbf{y}$,

$$\langle \mathcal{S}\mathbf{y}, (\mathcal{Q} + \mathcal{U}^*\mathcal{X}\mathcal{V} + \mathcal{V}^*\mathcal{X}^*\mathcal{U})\mathcal{S}\mathbf{y}\rangle \leq 0$$

for some $\mathbf{y} \in \mathbb{R}L_2^{k,l}$, implying

$$\langle \mathbf{y}, (\mathcal{S})^*\mathcal{Q}\mathcal{S}\mathbf{y}\rangle + \langle \mathbf{y}, (\mathcal{S})^*\mathcal{U}^*\mathcal{X}(\mathcal{V}\mathcal{S})\mathbf{y}\rangle + \langle \mathbf{y}, (\mathcal{V}\mathcal{S})^*\mathcal{X}^*\mathcal{U}\mathcal{S}\mathbf{y}\rangle \leq 0.$$

But $\mathcal{V}\mathcal{S}\mathbf{y} = \mathcal{V}(\mathcal{S}\mathbf{y}) = 0$, yielding

$$\langle \mathbf{y}, (\mathcal{S})^*\mathcal{Q}\mathcal{S}\mathbf{y}\rangle \leq 0,$$

for all $\mathbf{y} \in \mathbb{R}L_2^{k,l}$, resulting in the first inequality of Eq. (14).

Similarly, making $\mathbf{x} = \mathcal{R}\mathbf{v}$ for some $\mathbf{v} \in \mathbb{R}L_2^{i,j}$ yields the second inequality of Eq. (14). ■

B. Stabilizing Static Output Feedback Controller

In the following theorem, a sufficient condition for which the closed-loop PIE Eq. (9) is exponentially stable is presented. The optimization problem is convex if a controller gain is given in the form of 4-PI operator $\mathcal{K}$, as the solution of the LPI in Lem. 8.

Theorem 12: Consider 4-PI operators $\mathcal{A}, \mathcal{B}, \mathcal{C}$, and $\mathcal{K}$. If there are matrices $F \in \mathbb{R}^{n_u \times n_u}$ and $Z \in \mathbb{R}^{n_u \times n_y}$, constants $\alpha, \delta, \epsilon > 0$, and a self-adjoint 4-PI operator $\mathcal{P} \in \mathbf{\Pi}_{n,n}^{m,m}$, such that $\mathcal{P} \succcurlyeq \delta I$ and

$$\begin{bmatrix} -F - F^T + \epsilon I & \mathcal{B}^*\mathcal{P}\mathcal{T} + Z\mathcal{C} - F\mathcal{K} \\ \mathcal{T}^*\mathcal{P}\mathcal{B} + \mathcal{C}^* Z^T - \mathcal{K}^* F^T & \mathrm{He}\{\mathcal{T}^*\mathcal{P}\mathcal{A}_0\} + 2\alpha\mathcal{T}^*\mathcal{P}\mathcal{T} \end{bmatrix} \preccurlyeq 0, \quad (15)$$

with $\mathcal{A}_0 = \mathcal{A} + \mathcal{B}\mathcal{K}$, then the closed-loop PIE Eq. (8) defined by $\{\mathcal{A}, \mathcal{B}, \mathcal{K}\}$ and the closed-loop PIE Eq. (9) defined by $\{\mathcal{A}, \mathcal{B}, \mathcal{C}, L\}$, where $L = F^{-1}Z$, are both exponentially stable with decay rate α.

Proof: Our first goal is to show the invertibility of matrix F. Note that, from Eq. (15), $F + F^T \succcurlyeq \epsilon I \succ 0$. Then $x^T(F + F^T)x > 0$, for all $x \in \mathbb{R}^{n_u}$. But $x^T(F + F^T)x = 2x^T F x$, yielding $x^T F x > 0$. Next, from Cauchy-Shwartz inequality, $\|x\|\,\|Fx\| \geq |x^T F x| \geq x^T F x$. Consequently, $\|Fx\| > 0$, implying that F is non-singular.

Our task is now to show that the LPI Eq. (15) implies exponential stability of both closed-loop PIEs Eq. (8) and Eq. (9) with $L = F^{-1}Z$. The key observation is that, substituting the matrix $Z = FL$, it is clear that Eq. (15) can be rewritten as

$$\mathcal{Q} + \mathcal{U}^* F \mathcal{V} + \mathcal{V}^* F^* \mathcal{U} \preccurlyeq 0, \quad (16)$$

where,

$$\mathcal{Q} := \begin{bmatrix} \epsilon I & \mathcal{B}^*\mathcal{P}\mathcal{T} \\ \mathcal{T}^*\mathcal{P}\mathcal{B} & \mathrm{He}\{\mathcal{T}^*\mathcal{P}\mathcal{A}_0\} + 2\alpha\mathcal{T}^*\mathcal{P}\mathcal{T} \end{bmatrix},$$

$$\mathcal{V} = \begin{bmatrix}-I_{n_u} & L\mathcal{C}-\mathcal{K}\end{bmatrix} \in \mathbf{\Pi}_{0,mK}^{n_u,n_u+m},$$

$$\mathcal{U} = \begin{bmatrix}I_{n_u} & 0\end{bmatrix} := \mathit{\Pi}\left[\begin{array}{c|c}\begin{bmatrix}I_{n_u} & 0\end{bmatrix} & 0 \\ \hline \emptyset & \{\emptyset\}\end{array}\right] \in \mathbf{\Pi}_{0,mK}^{n_u,n_u+m},$$

I_{n_u} is the identity matrix in $\mathbb{R}^{n_u\times n_u}$ and 0 represents zero matrices of appropriate dimensions.

Now, Take the operator

$$\mathcal{R} = \begin{bmatrix}0\\ I_{mK}^m\end{bmatrix} := \mathit{\Pi}\left[\begin{array}{c|c}\begin{bmatrix}0\\ I_m\end{bmatrix} & 0 \\ \hline 0 & \{I,0,0\}\end{array}\right] \in \mathbf{\Pi}_{mK,mK}^{n_u+m,m},$$

where I_{mK}^m is the identity 4-PI operator in $\mathbf{\Pi}_{mK,mK}^{m,m}$. Note that $\mathcal{R}\mathbf{x} = \begin{bmatrix}0\\ \mathbf{x}\end{bmatrix} \in \mathbb{R}L_2^{n_u+m,mK}$, for all $\mathbf{x}\in\mathbb{R}L_2^{m,mK}$. Thus

$$\mathcal{U}(\mathcal{R}\mathbf{x}) = \mathit{\Pi}\left[\begin{array}{c|c}\begin{bmatrix}I_{n_u} & 0\end{bmatrix} & 0 \\ \hline \emptyset & \{\emptyset\}\end{array}\right]\begin{bmatrix}0\\ \mathbf{x}\end{bmatrix} = 0 \in \mathbb{R}^{n_u}.$$

Moreover, $\mathcal{R}^*\mathcal{R} = I_{mK}^m \succ 0$. Then, $\mathcal{R}$ is a right annihilator of $\mathcal{U}$. Similarly, consider

$$\mathcal{S} = \begin{bmatrix}L\mathcal{C}-\mathcal{K}\\ I_{mK}^m\end{bmatrix} \in \mathbf{\Pi}_{mK,mK}^{n_u+m,m},$$

then

$$\mathcal{V}(\mathcal{S}\mathbf{x}) = -(L\mathcal{C}-\mathcal{K})\mathbf{x} + (L\mathcal{C}-\mathcal{K})\mathbf{x} = 0 \in \mathbb{R}^{n_u},$$

for all $\mathbf{x}\in\mathbb{R}L_2^{n_u+m,mK}$ and $\mathcal{S}$ is a right annihilator of $\mathcal{V}$.

Thus, from Lem. 11, Eq. (16) implies two inequalities. The first one, $(\mathcal{R})^*\mathcal{Q}\mathcal{R} \preccurlyeq 0$ implies

$$(\mathcal{A}+\mathcal{B}\mathcal{K})^*\mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}(\mathcal{A}+\mathcal{B}\mathcal{K}) \preccurlyeq -2\alpha\mathcal{T}^*\mathcal{P}\mathcal{T},$$

which provides a stability certificate for the closed-loop PIE Eq. (8) according to Lem. 7, using the Lyapunov functional $V(\mathcal{T}\mathbf{x}(t)) = \langle\mathcal{T}\mathbf{x}(t), \mathcal{P}\mathcal{T}\mathbf{x}(t)\rangle$. On the other hand, $(\mathcal{S})^*\mathcal{Q}\mathcal{S} \preccurlyeq 0$ implies

$$(\mathcal{A}^* + \mathcal{C}^*L^T\mathcal{B}^*)\mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}(\mathcal{A}+\mathcal{B}L\mathcal{C}) \preccurlyeq -2\alpha\mathcal{T}^*\mathcal{P}\mathcal{T},$$

Again, by Lem. 7, this provides a stability certificate for the closed-loop PIE Eq. (9) using the same Lyapunov functional. ∎

Remark 2: Note that for the condition in Thm. 12, the full-state feedback controller $\mathcal{K}$ must be given, linearizing the otherwise bilinear inequality. Thus, we have an LPI in Eq. (15) with the necessary condition that the closed-loop PIE under full-state feedback Eq. (8), defined by $\{\mathcal{A},\mathcal{T},\mathcal{B},\mathcal{K}\}$, is exponentially stable. Such a $\mathcal{K}$ can be found using the LPI in Sec. III-D as a first-step and using the computed $\mathcal{K}$ as an input to Thm. 12. Thus, we have a two-step procedure based on solving an LPI in each step. This procedure is implemented in the numerical simulations of Sec. V.

Remark 3: In contrast with the previous approach to SOF of delay systems presented in [12], we use a full-state feedback controller in the first stage that considers the history of the state. Specifically, the solution of Lem. 8, if exists, gives a control law of the type $u(t) = K_1x(t) + \int_{-1}^0 K_2(s)\partial_s x(t+s\tau)ds$ in the case of a single delay τ, where K_1 is a matrix and K_2 is a polynomial matrix in s. This more general first stage allow us to obtain less conservative results as shown in Sec. V.

V. Numerical Examples

In this section, we validate the proposed algorithm for controller synthesis by constructing controller gains $\mathcal{K}$ and L sequentially for unstable delay systems and simulating closed-loop dynamics subject to non-zero initial conditions. The LPIs of Lem. 8 and Thm. 12 were computed using the Matlab toolbox PIETOOLS [22] and the convex-optimization solver MOSEK, for the values of $\delta = 10^{-6}$, and $\epsilon = 10^{-4}$. For the simulations, a time step of $0.01s$ was used in PIESIM [23], a numerical simulator of PIEs integrated with PIETOOLS.

Example 1: The following system was considered in [4]:

$$\dot{x}(t) = \begin{bmatrix}0 & 1 & 0 & 0\\ \frac{-2k}{M} & 0 & \frac{-mg}{M} & 0\\ 0 & 0 & 0 & 1\\ \frac{2k}{Ml} & 0 & \frac{(m+M)g}{Ml} & 0\end{bmatrix}x(t) + \begin{bmatrix}0\\ \frac{1}{M}\\ 0\\ \frac{-1}{Ml}\end{bmatrix}u(t), \quad (17)$$

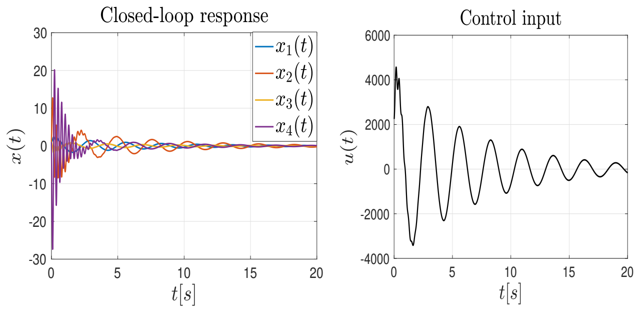
$$y(t) = \begin{bmatrix}1&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix}x(t) + \begin{bmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&0&1&0\end{bmatrix}x(t-\tau),$$

where we adopt the same values of the reference for the parameters and $\tau = 0.1s$. The PIE parametrization of this system can be computed by using Fig. (7), and then the LPI from Lem. 8 can be solved using PIETOOLS. Finaly, the LPI in Thm. 12 provides the SOF gain. A bisection algorithm can be used to maximize the decay rate α by solving Lem. (8) and Thm. 12 in each iteration. Running the bisection algorithm, one can obtain the controller $u(t) = \begin{bmatrix}2374.12 & 321.31 & -317.25 & -209.37\end{bmatrix}y(t)$. The state trajectories of the resulting closed-loop system with initial condition $x(t) = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^T$, for all $t\in[-0.1,0]$, are presented in Fig. 2.

In [4], a non-convex numerical optimization was performed, depending on initial starting points, to find a stabilizing controller maximizing the decay rate α. For this problem, comparatively, the obtained values of control gain and spectral abscissa with TDS-CONTROL are $10^3\begin{bmatrix}8.11 & 4.99 & -5.71 & -2.48\end{bmatrix}$ and -1.4059. Using TDS-CONTROL to verify the spectral abscissa with the controller derived here, the obtained value is -2.2732. Note from Fig. 2 that the control input is high due to the maximization of α, but the resultant spectral abscissa is further away from the imaginary axis than in the closed-loop system obtained with TDS-CONTROL, even though the controller gain has smaller values.

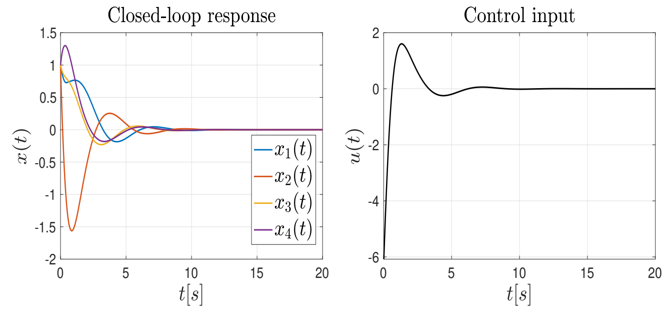
Example 2: The following example was considered in [12]:

$$\dot{x}(t) = \begin{bmatrix}0&0&1&0\\0&0&0&1\\-1&1&0&\\1&-1&0&0\end{bmatrix}x(t) \quad (18)$$
$$+\begin{bmatrix}0&0&0.1&0\\0&0&0&0.1\\-0.1&0.1&0&0\\0.1&-0.1&0&0\end{bmatrix}x(t-\tau) + \begin{bmatrix}0\\0\\1\\0\end{bmatrix}u(t),$$



(a) (b)

Fig. 2: (a) Trajectories of the states of the closed-loop system in Eq. (17) under non-zero initial conditions $x(t) = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^T$, for all $t \in [-0.1, 0]$. (b) The corresponding control input $u(t)$.



(a) (b)

Fig. 3: (a) Trajectories of the states of the closed-loop system in Eq. (19) under non-zero initial conditions $x(t) = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^T$, for all $t \in [-0.45, 0]$. (b) The corresponding control input $u(t)$.

$$y(t) = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix} x(t).$$

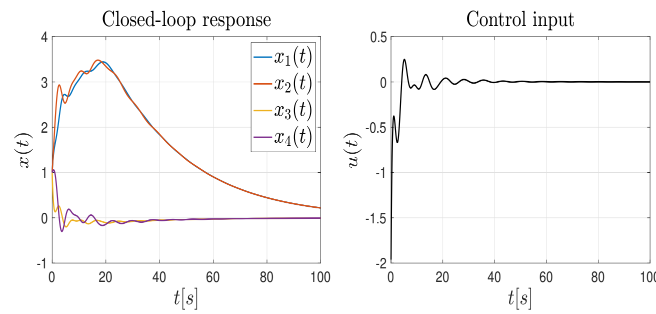
The LPI conditions of Lem. (8) and Thm. (12) were implemented with $\alpha = 10^{-12}$, and the SOF control law was obtained as $u(t) = \begin{bmatrix}-0.055832 & -1.9481\end{bmatrix} y(t)$ for $\tau = 20s$. The state trajectories with initial condition $x(t) = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^T$, for all $t \in [-20, 0]$ are presented in Fig. 4. Notably, the method proposed herein guarantees feasible solutions for arbitrary delay values, demonstrating its low conservatism.

Comparatively, the two-stage solution proposed in [12] can be adapted to only require the stability of the closed-loop system, but fails to give a feasible solution unless an iterative algorithm to optimize the gain of the first stage is used. Since the solution of [12] is delay-independent, we know that the system is stabilizable independently of the value of the delay, as our result suggests. A key limitation of the first stage in [12] is to not consider a feedback control proportional to the full infinite-dimensional state of the delay system, as Lem. (8) does.

Example 3: The following example was considered in [7]:

$$\dot{x}(t) = \begin{bmatrix}-0.2 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ -1 & 0 & -1 & 1\\ 0 & 1 & 1 & 0\end{bmatrix} x(t) \tag{19}$$
$$+ \begin{bmatrix}-0.8 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} x(t-\tau) + \begin{bmatrix}0\\ 1\\ 0\\ 0\end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix} x(t).$$

The LPI conditions of Lem. (8) and Thm. (12) were implemented with $\alpha = 10^{-12}$, and the SOF control was obtained as $u(t) = \begin{bmatrix}-2.8216 & -3.392\end{bmatrix} y(t)$ for $\tau = 0.45$. The state trajectories with initial condition $x(t) = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^T$, for all $t \in [-0.45, 0]$ are presented in Fig. 3. It is possible to obtain feasible results for the corresponding system with a delay value τ up to $1.12s$. Comparatively, applying the adapted LMIs of [12] to this system, a feasible solution cannot be obtained, highlighting the reduction in conservatism of the new procedure.



(a) (b)

Fig. 4: (a) Trajectories of the states of the closed-loop system in Eq. (18) under non-zero initial conditions $x(t) = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^T$, for all $t \in [-20, 0]$. (b) The corresponding control input $u(t)$.

Example 4: The following example was adapted from [24]:

$$\dot{x}(t) = \begin{bmatrix}-1 & 2\\ 0 & 1\end{bmatrix} x(t) + \begin{bmatrix}0.6 & -0.4\\ 0 & 0\end{bmatrix} x(t-\tau_1) \tag{20}$$
$$+ \begin{bmatrix}0 & 0\\ 0 & -0.5\end{bmatrix} x(t-\tau_2) + \begin{bmatrix}0\\ 1\end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix}0 & 1\end{bmatrix} x(t).$$

The LPI conditions of Lem. (8) and Thm. (12) were implemented with $\alpha = 10^{-12}$, and the SOF control law was obtained as $u(t) = -6.792 y(t)$ for $\tau_1 = 1s$ and $\tau_2 = 2s$. The state trajectories with initial condition $x(t) = \begin{bmatrix}1 & 1\end{bmatrix}^T$, for all $t \in [-2, 0]$ are presented in Fig. 5.

VI. Conclusion

This work derives a procedure to compute a stabilizing SOF controller for systems with multiple delays in the state and output, which for the best of the author's knowledge is not possible with the available LMI-based solutions to the SOF problem. This achievement is based on extending a previous approach used in ODEs where the same Lyapunov

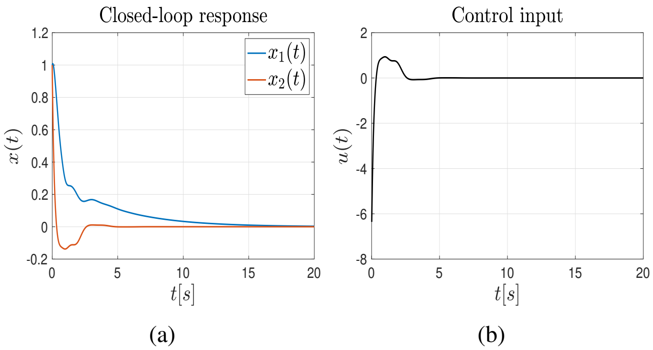


Fig. 5: (a) Trajectories of the states of the closed-loop system in Eq. (20) under non-zero initial conditions $x(t) = \begin{bmatrix}1 & 1\end{bmatrix}^T$, for all $t \in [-2, 0]$. (b) The corresponding control input $u(t)$.

function is required to prove stability of both the closed-loop systems under full-state feedback and SOF. The delay system is represented as a PIE defined by partial integral operators with polynomial kernel. The algebraic parametrization of PIEs allows the formulation of convex optimization problems with little conservatism and no discretization.

Then, the Projection Lemma is extended to the algebra of 4-PI operators and leveraged to derive a bilinear condition for SOF that unifies the state and output feedback problems in one inequality. In the derived condition, the bilinearity is due to the gain of the full-state feedback controller. To circumvent the bilinearity, the full-state feedback controller is computed based on previous results in PIEs and then used as an input to the main problem, which can then be solved with existing software. Finally, the numerical validation with examples from the literature highlight the reduction in conservatism we acquire by leveraging the PIEs framework.

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