A Convex Approach for Stability Analysis of Partial Differential Equations

by

Evgeny Meyer

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Graduate Supervisory Committee:

Matthew Peet, Chair
Spring Berman
Daniel Rivera

ARIZONA STATE UNIVERSITY

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ABSTRACT

This thesis presents a computational framework based on convex optimization for stability analysis of systems described by Partial Differential Equations (PDEs). Specifically, two forms of linear PDEs with spatially dependent polynomial coefficients are considered.

Firstly, a class of coupled PDEs with one spatial variable is considered. Parabolic, elliptic or hyperbolic PDEs with Dirichlet, Neumann or mixed boundary conditions can be reformulated in order to be used by the framework. As an example, the reformulation is presented for systems governed by Schrödinger equation, parabolic type, and acoustic wave equation, hyperbolic type. The second form of PDEs of interest are scalar-valued with two spatial variables. An extra spatial variable allows consideration of problems such as local stability of fluid flow in channels and dynamics of population over two dimensional domains.

The approach does not involve discretization and is based on using Sum-of-Squares (SOS) polynomials and positive semi-definite matrices to parameterize operators which are positive on function spaces. Applying the parameterization to construct Lyapunov functionals with negative derivatives allows to express stability conditions as a set of Linear Matrix Inequalities (LMIs). The MATLAB package SOSTOOLS was used to construct the LMIs. The resultant LMIs then can be solved using existent Semi-Definite Programming (SDP) solvers such as SeDuMi or MOSEK. Moreover, the proposed approach allows to calculate bounds on the rate of decay of the solution norm.

The methodology is tested using several numerical examples and compared with the results obtained from simulation using standard methods of numerical discretization and analytic solutions.

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RESEARCH GOALS AND MOTIVATION

Partial Differential Equations (PDEs) are often used to model systems in which the quantity of interest varies continuously in both space and time. Examples of such quantities include: deflection of beams (Euler-Bernoulli equation); velocity and pressure of fluid flow (Navier-Stokes equations); and population density in predator-prey models. See Evans (1998), Garabedian (1964) and John (1982) for a wide range of examples.

Stability analysis and controller design for PDEs is an active area of research Christofides (2012a), Curtain and Zwart (1995). One approach to stability analysis of PDEs is to approximate the PDEs with Ordinary Differential Equations (ODEs) using, e.g. Galerkin's method or finite difference, and then apply finite-dimensional optimal control methods, e.g. El-Farra *et al.* (2003), Baker and Christofides (2000), Kamyar *et al.* (2013). We present a methodology for stability analysis without discretization. Specifically, we use Linear Matrix Inequalities (LMIs) and Sum-of-Squares (SOS) optimization to construct Lyapunov functionals for PDEs with spatially dependent coefficients.

It is well-known that existence of a Lyapunov function for a system of ODEs or PDEs is a sufficient condition for stability. For example, Fridman and Orlov (2009) uses a Lyapunov approach and Linear Operator Inequalities (LOIs) to provide sufficient conditions for exponential stability of a controlled heat and delayed wave equations. Another method based on Lyapunov and semigroup theories was applied in Fridman *et al.* (2010) for analysis of wave and beam PDEs with constant coefficients and delayed boundary control. In Solomon and Fridman (2015) a Lyapunov based analysis of semilinear diffusion equations with delays gave stability conditions in terms of LMIs. In Semigroup Theory the state of the system belongs to a certain space of functions. The solution map for these systems is an operator-valued function ("strongly continuous semigroup" - SCS), indexed to the time domain, which maps the current state to a future state. For an introduction to Semigroup Theory we refer readers to Lasiecka (1980), Curtain and Zwart (1995).

In the semigroup framework, stability, controllability and observability conditions can be expressed using operator inequalities in the same way that LMIs are used to represent those properties for ODEs. As an example, for a system $\dot{u} = \mathcal{A}u$ which defines a SCS on a Hilbert space X with $\mathcal{A}$ being the infinitesimal generator, the exponential stability of the system is equivalent to the existence of a positive bounded linear operator $\mathcal{P} : X \to X$ such that

$$\langle u, \mathcal{A}\mathcal{P}u \rangle_X + \langle \mathcal{A}u, \mathcal{P}u \rangle_X \leq - \langle u, u \rangle_X \tag{1.1}$$

for all u in the domain of $\mathcal{A}$. Condition (1.1) is termed a Linear Operator Inequality (LOI). The terminology LOI is deliberately chosen to suggest a parallel to the use of Linear Matrix Inequalities (LMIs) for computational analysis and control of ODEs. Indeed, there have been efforts to use discretization to solve LOI type conditions for stability analysis and optimal control of PDEs (see, e.g. Christofides (2012b)), optimal actuator placement for parabolic PDEs (see Demetriou and Borggaard (2003) and Morris *et al.* (2015)). However, in this paper, we do not employ discretization. While discretization has proven quite effective in practice, one should note that in general it is difficult to determine if feasibility of the discretized LOI implies stability of the non-discretized PDE. In contrast, this paper is focused on exploring how to use computation to solve LOIs (1.1) directly by parameterizing the cone of positive and negative operators.

An alternative approach, taken by Peet (2014), uses some of the machinery developed for DDEs to express Lyapunov inequalities as LMIs, which can then be tested using standard interior-point algorithms. We also note that in Fridman and Terushkin (2016) stability analysis and initial state recovery of semi-linear wave equation are presented in terms of LMIs. Recently, our lab, in collaboration with other researchers have begun to explore how to use the SOS method for optimization of polynomials to study analyze and control PDEs without the need for discretization. Specifically, in Papachristodoulou and Peet (2006a), we considered stability analysis of scalar non-linear PDEs using a simple form of Lyapunov function. This simple Lyapunov function was recently extended in Valmorbida *et al.* (2014a) and in Valmorbida *et al.* (2015) to consider some forms of coupled PDEs and in Ahmadi *et al.* (2016) to consider passivity. In Gahlawat and Peet (2015) and related publications, the class of Lyapunov functions was expanded to squares of semi-separable integral operators and applied to output-feedback dynamic control of scalar PDEs. Finally, in Meyer and Peet (2015), we considered stability of PDEs with multiple spatial variables.

Extensive examples of applying the backstepping method to the boundary control of PDEs can be found in Krstic and Smyshlyaev (2008c), Krstic and Smyshlyaev (2008b), Krstic and Smyshlyaev (2008a), Smyshlyaev and Krstic (2005). Briefly speaking, backstepping uses a Volterra operator to search for an invertible mapping from the original PDE to a chosen "target" PDE, known to be stable. In order to find such mapping, one has to solve analytically or numerically a PDE for Volterra operator's kernel. If the mapping is found, it provides the boundary control law. Applications for two-dimensional cases were discussed in Vazquez and Krstic (2006) and Xu *et al.* (2008). However, backstepping requires us to guess on the target PDE and solve the PDE for kernel, which may be a challenging task for PDEs with two spatial variables and spatially dependent coefficients. Moreover, backstepping cannot be used for stability analysis in the absence of a control input.

Note that SOS has been applied in Papachristodoulou and Peet (2006b) and Valmorbida *et al.* (2014b) to find Lyapunov functionals for 1D parabolic PDE. Input-Output analysis of PDEs with SOS implementation is discussed in Ahmadi *et al.* (2014). Examples of using SOS in controller and observer designs for parabolic linear one-dimensional PDEs can be found in Gahlawat and Peet (2011), Gahlawat and Peet (2014), Gahlawat *et al.* (2011).

In Chapter 2 we introduce notations in use and some preliminaries. General Lyapunov theorem is proved in Chapter 3. Proposed techniques for coupled PDEs and PDEs with two spatial variables are presented in Chapters 4 and 5. Numerical results are discussed in Chapter 6 and we conclude the work in Chapter 7.

1.1 Formulation of Mathematical Problems

The interest of this work is to propose a computational algorithm for stability analysis of two following forms of PDEs.

1.1.1 Coupled linear PDEs with one spatial variable

First form considers function $u : [0, \infty) \times [a, b] \to \mathbb{R}^n$ satisfying

$$u_t(t,x) = A(x)u_{xx}(t,x) + B(x)u_x(t,x) + C(x)u(t,x) \tag{1.2}$$

for all $t > 0$ and $x \in (a, b)$, where $a, b \in \mathbb{R}$. The coefficients A, B, C are polynomial matrices. Boundary conditions are represented through the matrix $D \in \mathbb{R}^{4n\times 4n}$, i.e.

for all $t > 0$

$$D \begin{bmatrix} u(t,a) \\ u(t,b) \\ u_x(t,a) \\ u_x(t,b) \end{bmatrix} = 0. \tag{1.3}$$

Matrix D allows for different types of boundary conditions. In case of homogeneous Dirichlet boundary conditions

$$D = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Mixed boundary conditions, i.e., for example, homogeneous Neumann at $x = a$ and Dirichlet at $x = b$, can be written as (1.3) with

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution to (1.2) is assumed to exist, be unique and depend continuously on the initial condition $u(0, x)$. Also, for each $t > 0$ we suppose $u(t, \cdot), u_x(t, \cdot), u_{xx}(t, \cdot) \in L_2^n(a, b)$.

1.1.2 Parapolic scalar-valued PDEs with two spatial variables

The second form of interest has two spatial variables. Specifically, for all $t > 0$ and $x \in \Omega := (0, 1)^2$, $u : [0, \infty) \times \Omega \to \mathbb{R}$ satisfies

$$\begin{aligned} u_t(t,x) = a(x)u_{x_1x_1}(t,x) + b(x)u_{x_1x_2}(t,x) + c(x)u_{x_2x_2}(t,x) + d(x)u_{x_1}(t,x) \\ + e(x)u_{x_2}(t,x) + f(x)u(t,x), \end{aligned} \tag{1.4}$$

where a, b, c, d, e, f are polynomials. As before, assume that solution to (1.4) exists, is unique and depends continuously on initial conditions. Moreover, u satisfies zero Dirichlet boundary conditions, i.e.

$$u(t,1,x_2) = 0,\ u(t,0,x_2) = 0,\ u(t,x_1,1) = 0,\ u(t,x_1,0) = 0$$

for all $x_1, x_2 \in [0,1]$ and $t \geq 0$.

In the following sections it is shown how some "well-known" PDEs can be formulated as (1.2).

1.2 Example 1: Schrödinger Equation

To illustrate the class of PDEs which can be written as (1.2), first consider the Schrodinger equation. In the following equation V is the potential energy, i is the imaginary unit, $\hbar$ is the reduced Planck constant and ψ is the wave function of the quantum system.

$$i\hbar\psi_t(t,x) = -\frac{\hbar^2}{m}\psi_{xx}(t,x) + V(x)\psi(t,x)$$

can be written as two coupled PDEs if one decomposes the solution into real and imaginary parts as $\psi(t,x) = \psi^{rl}(t,x) + i\psi^{im}(t,x)$ and then separates the real and imaginary parts of the equation, i.e.

$$\begin{bmatrix} \psi_t^{rl}(t,x) \\ \psi_t^{im}(t,x) \end{bmatrix} = \underbrace{\frac{\hbar}{m}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} \psi_{xx}^{rl}(t,x) \\ \psi_{xx}^{im}(t,x) \end{bmatrix} + \underbrace{\frac{V(x)}{\hbar}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{C(x)} \begin{bmatrix} \psi^{rl}(t,x) \\ \psi^{im}(t,x) \end{bmatrix}.$$

1.3 Example 2: Model for Acoustic Waves

Next consider a model for a 1-D acoustic wave. For all $t > 0, r \in (0,R)$ and some fixed $c > 0$,

$$p_{tt}(t,r) = c^2 p_{rr}(t,r) + \frac{2c^2}{r}p_r(t,r). \tag{1.5}$$

PDE (1.5) is equivalent to a system of two coupled first order PDEs as

$$\begin{bmatrix} q_t(t,r) \\ p_t(t,r) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & c^2 \\ 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} q_{rr}(t,r) \\ p_{rr}(t,r) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & \frac{2c^2}{r} \\ 0 & 0 \end{bmatrix}}_{B(r)} \begin{bmatrix} q_r(t,r) \\ p_r(t,r) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} q(t,r) \\ p(t,r) \end{bmatrix},$$

where q is an auxiliary function. Moreover, if the boundary conditions imply amplification of the waves, i.e.

$$p(t,0) = f_1 p(t,R) \quad \text{and} \quad p_r(t,0) = f_2 p_r(t,R)$$

for some $f_1, f_2 > 0$ and all $t > 0$, then the boundary conditions can be stated using (1.3) with

$$D = \begin{bmatrix} 0 & 1 & 0 & -f_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -f_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The next chapter presents notations we are using and some preliminaries.

Chapter 2

PRELIMINARIES

2.1 Notations

$\mathbb{N}$ is the set of natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. $\mathbb{R}^n$ and $\mathbb{S}^n$ are the n-dimensional Euclidean space and space of $n \times n$ real symmetric matrices. For $x \in \mathbb{R}^n$, let x^T denote transposed x and $x_i \in \mathbb{R}$ is the i-th component of x. $\|\cdot\|_1$ is a norm on $\mathbb{R}^n$, defined as $\|x\|_1 := \sum_{i=1}^n |x_i|$. For $X \in \mathbb{S}^n$, $X \leq 0$ means that X is negative semidefinite. The symbol $*$ will denote the symmetric elements of a symmetric matrix.

For $\Omega \subseteq \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ let $f(x)$ stand for $f(x_1, ..., x_n)$ and $\int_\Omega f(x)\, dx$ represent an integral of f over Ω with $dx := dx_1 dx_2 ... dx_n$.

Let $\mathbb{N}_0^n := \{\alpha \in \mathbb{R}^n : \alpha_i \in \mathbb{N}_0\}$. A vector $\alpha \in \mathbb{N}_0^n$ is called multi-index. For $l \in \mathbb{N}$ define the set

$$Q_l^n := \{\alpha \in \mathbb{N}_0^n : \|\alpha\|_1 \leq l\}. \tag{2.1}$$

For $\alpha \in \mathbb{N}_0^n$, $x \in \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ partial derivative

$$D^\alpha[g(x)] := \frac{\partial^\alpha}{\partial x^\alpha}[g(x)] = \prod_{i=1}^n \frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}}[g(x)]. \tag{2.2}$$

Note that $\frac{\partial^0}{\partial x_i^0}[g(x)] = g(x)$ for any $i \in \{1, ..., n\}$. Classical notations such as $u_{x_1 x_2}(t, x) := \frac{\partial}{\partial x_2}[\frac{\partial}{\partial x_1}[u(t, x)]]$ are also applied.

If for a function $f : \Omega \to \mathbb{R}$ and some $\alpha \in \mathbb{N}_0^n$ derivative $D^\alpha[f(x)]$ exists for all $x \in \Omega$, there exists $g : \Omega \to \mathbb{R}$ such that $g(x) = D^\alpha[f(x)]$ for all $x \in \Omega$. For brevity $D^\alpha[f] := g$.

$L_p(\Omega)$ stands for the space of Lebesgue-measurable functions $g : \Omega \to \mathbb{R}$ with norm, for $p \in \mathbb{N}$

$$\|g\|_{L_p} := \Big(\int_\Omega |g(s)|^p ds \Big)^{1/p}$$

and $\|g\|_{L_\infty} := \sup_{s \in \Omega} |g(s)|$. Note that if $g : \Omega \to \mathbb{R}^m$ then the notation $L_p^m(\Omega)$ is used.

$W^{k,p}(\Omega)$ denotes Sobolev space of functions $u : \Omega \to \mathbb{R}$ with $D^\alpha[u] \in L_p(\Omega)$ for all $\alpha \in Q_k^n$, where Q_k^n is defined as in (2.1) and norm

$$\|u\|_{k,p} := \sum_{\|\alpha\|_1 \le k} \|D^\alpha[u]\|_{L_p}.$$

It is known that for continuous functions $u : [0, \infty) \to W^{2,2}(\Omega)$ and $V : W^{2,2}(\Omega) \to \mathbb{R}$ the composition $(V \circ u) : [0, \infty) \to \mathbb{R}$ is also continuous and the upper right-hand derivative $D_t^+ V(u(t))$ is defined by

$$D^+[V(u(t))] := \limsup_{h \to 0^+} \frac{V(u(t+h)) - V(u(t))}{h}.$$

Note that if $v : [0, \infty) \to \mathbb{R}$ is differentiable at $t \in (0, \infty)$ then $D^+[v(t)] = \frac{d}{dt}[v(t)]$.

2.2 Linear Matrix Inequalities

Firstly, let start with the general form of an SDP. For some $c \in \mathbb{R}^n, b \in \mathbb{R}^k, A \in \mathbb{R}^{k \times n}$ and $F \in \mathbb{S}^m$

$$\begin{aligned} &\min_{x \in \mathbb{R}^n} c^T x \\ &\text{such that } F_0 + \sum_{i=1}^n x_i F_i \le 0 \end{aligned}$$

SDPs are convex optimization problems and, thus, can be solved using interior point method.

The feasibility problem of an SDP is known as an LMI. Finite number of LMIs can be cast as single LMI. The problem of searching for an $X \in \mathbb{S}^n$ such that

$$X > 0 \quad \text{and} \quad A^T X + XA < 0 \tag{2.3}$$

where $A \in \mathbb{R}^{n\times n}$ is given, can be cast as LMI. As an example, let $n = 2$ and denote

$$X = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix}. \tag{2.4}$$

If $e_{ij} \in \mathbb{R}^{2\times 2}$ are matrices with $e(i,j) = 1$ and zero other elements, then (2.4) can be written as

$$X = x_1 e_{11} + x_2 e_{12} + x_2 e_{21} + x_3 e_{22}. \tag{2.5}$$

and, therefore, problem (2.3) can be cast as

$$F_0 + \sum_{i=1}^{3} x_i F_i \leq 0$$

with

$$F_0 = \begin{bmatrix} \epsilon I_2 & 0 \\ 0 & \epsilon I_2 \end{bmatrix}, \; F_1 = \begin{bmatrix} -e_{11} & 0 \\ 0 & A^T e_{22} + e_{22} A \end{bmatrix},$$

$$F_2 = \begin{bmatrix} -(e_{12} + e_{21}) & 0 \\ 0 & A^T(e_{12} + e_{21}) + (e_{12} + e_{21})A \end{bmatrix},$$

$$F_3 = \begin{bmatrix} -e_{22} & 0 \\ 0 & A^T e_{22} + e_{22} A \end{bmatrix}.$$

Thus, one can solve (2.5) using any SDP solver and that will provide a solution to (2.3).

2.3 Polynomials and Sum of Squares Polynomials

For a multi-index $\alpha \in \mathbb{N}_0^n$ and $x \in \mathbb{R}^n$, let

$$x^\alpha := \prod_{i=1}^{n} x_i^{\alpha_i} = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}.$$

Then x^α is a monomial of degree $\|\alpha\|_1 \in \mathbb{N}_0$. A polynomial is a finite linear combination of monomials $p(x) := \sum_\alpha p_\alpha x^\alpha$, where the summation is applied over a given finite set of multi-indexes α and $p_\alpha \in \mathbb{R}$ denotes the corresponding coefficient. The degree of a polynomial p is the largest degree among all monomials, and is denoted by $\deg(p) \in \mathbb{N}_0$.

A polynomial p is called Sum of Squares (SOS), if there is a finite number of polynomials z_i such that for all $x \in \mathbb{R}^n$, $p(x) = \sum_i q_i(x)^2$. If p is a SOS polynomial, then $p(x) \geq 0$ for all $x \in \mathbb{R}^n$.

Polynomial matrices and SOS polynomial matrices are defined in a similar manner, except p_α are not scalars, but matrices. If M is a SOS polynomial matrix then for all $x \in \mathbb{R}^n$, $M(x) \geq 0$.

The following theorem introduces the connection between SOS polynomials and positive semi-definite matrices. For more see Parrilo (2000).

Theorem 1. *A polynomial $p : \mathbb{R}^n \to \mathbb{R}$ of degree $2d$ is an SOS polynomial if and only if there exists $Q \in \mathbb{S}^{d+1}$ such that $Q \geq 0$ and*

$$p(x) = z_d^T(x) Q z_d(x),$$

where $z_d(x)$ is a vector of monomials up to degree d, i.e.

$$z_d(x) := \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^d \end{bmatrix}.$$

Proof. ($\Rightarrow$) Suppose p is an SOS polynomial. Then there are polynomials q_i such that

$$p(x) = \sum_{i=1}^{k} q_i(x)^2.$$

Note, that for each $i \in \{1, ..., k\}$,

$$q_i(x) = a_i^T z_d(x),$$

where a_i is the vector of coefficients of the polynomial q_i. Then

$$\begin{aligned}
\sum_{i=1}^{k} q_i(x)^2 &= \begin{bmatrix} q_1(x) \\ q_2(x) \\ \vdots \\ q_k(x) \end{bmatrix}^T \begin{bmatrix} q_1(x) \\ q_2(x) \\ \vdots \\ q_k(x) \end{bmatrix} = \begin{bmatrix} a_1^T z_d(x) \\ a_2^T z_d(x) \\ \vdots \\ a_k^T z_d(x) \end{bmatrix}^T \begin{bmatrix} a_1^T z_d(x) \\ a_2^T z_d(x) \\ \vdots \\ a_k^T z_d(x) \end{bmatrix} \\
&= z_d(x)^T \begin{bmatrix} a_1 & a_2 & \dots & a_k \end{bmatrix} \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_k^T \end{bmatrix} z_d(x) = z_d(x)^T A A^T z_d(x) \\
&= z_d(x)^T Q z_d(x).
\end{aligned}$$

Since $Q = AA^T$, $Q \geq 0$.

($\Leftarrow$) Given polynomial p suppose there exists $Q \geq 0$ such that

$$p(x) = z_d(x)^T Q z_d(x). \tag{2.6}$$

Since $Q \geq 0$ there exists A such that $Q = A^T A$. Then (2.6) can be reformulated as

$$p(x) = z_d(x)^T A^T A z_d(x) = (A z_d(x))^T A z_d(x). \tag{2.7}$$

Let $q(x) := A z_d(x)$, then (2.7) can be continued as

$$p(x) = q(x)^T q(x) = \begin{bmatrix} q_1(x) \\ q_2(x) \\ \vdots \\ q_k(x) \end{bmatrix}^T \begin{bmatrix} q_1(x) \\ q_2(x) \\ \vdots \\ q_k(x) \end{bmatrix} = \sum_{i=1}^{k} q_i(x)^2.$$

$\square$

2.4 Comparison Principe

Recall the Comparison Lemma which is used in the proof of Lyapunov theorem for PDEs. Verbatim from Khalil and Grizzle (1996).

Lemma 1. *Consider the scalar differential equation* $\frac{d}{dt}[u(t)] = f(t, u(t))$, $u(t_0) = u_0$, *where* $f(t, x)$ *is continuous in* t *and locally Lipschitz in* x, *for all* $t \geq 0$ *and all* $x \in J \subset \mathbb{R}$. *Let* $[t_0, T)$ *(*T *could be infinity) be the maximal interval of existence of the solution* u, *and suppose* $u(t) \in J$ *for all* $t \in [t_0, T)$. *Let* v *be a continuous function whose upper right-hand derivative* $D^+[v(t)]$ *satisfies*

$$D^+[v(t)] \leq f(t, v(t)), \quad v(t_0) \leq u_0$$

with $v(t) \in J$ *for all* $t \in [t_0, T)$. *Then,* $v(t) \leq u(t)$ *for all* $t \in [t_0, T)$.

Proof. For the proof see Khalil and Grizzle (1996). $\square$

Chapter 3

LYAPUNOV TEST FOR PDES

In this chapter Lyapunov conditions for stability are presented for the following general class of parabolic PDEs. For some $k \in \mathbb{N}$, all $t \in (0, \infty)$ and $x \in \Omega \subseteq \mathbb{R}^n$,

$$u_t(t,x) = f(t, x, D^{\alpha(1)}\left[u(t,x)\right], ..., D^{\alpha(k)}\left[u(t,x)\right]), \tag{3.1}$$

where $u : [0, \infty) \times \Omega \to \mathbb{R}^m$ and for each $i \in \{1, ..., k\}$, $D^{\alpha(i)}[u(t,x)]$ is a partial derivative in x. Assume that solutions to (3.1) exist, are unique and depend continuously on initial conditions.

Definition 1. *The PDE (3.1) is called exponentially stable in the sense of L_2 if there exist scalars $k, \alpha > 0$ such that*

$$\|u(t,\cdot)\|_{L_2^m} \leq k e^{-\alpha t} \quad \text{for all } t > 0.$$

The following theorem provides sufficient conditions for (3.1) to be exponentially stable.

Theorem 2. *Let there exist continuous $V : L_2^m(\Omega) \to \mathbb{R}$, $l, p \in \mathbb{N}$ and $b, a > 0$ such that*

$$a\|w\|_{L_2^m}^l \leq V(w) \leq b\|w\|_{L_2^m}^p, \tag{3.2}$$

for all $w \in L_2^m(\Omega)$. Furthermore, suppose that there exists $c \geq 0$ such that for all $t \geq 0$ the upper right-hand derivative

$$D^+\left[V(u(t,\cdot))\right] \leq -c\|u(t,\cdot)\|_{L_2}^p, \tag{3.3}$$

where u satisfies (3.1). Then for all $t \geq 0$

$$\|u(t,\cdot)\|_{L_2} \leq \sqrt[l]{\frac{b}{a}}\|u(0,\cdot)\|_{L_2}^{p/l} \exp\left\{-\frac{c}{lb}t\right\}.$$

Proof. Let conditions of Theorem 2 be satisfied. From (3.2) it follows that for each $t \geq 0$

$$a\|u(t,\cdot)\|^l_{L_2^m} \leq V(u(t,\cdot)) \leq b\|u(t,\cdot)\|^p_{L_2^m}. \tag{3.4}$$

Dividing both sides of the second inequality in (3.4) by b results in

$$\frac{1}{b}V(u(t,\cdot)) \leq \|u(t,\cdot)\|^p_{L_2^m}. \tag{3.5}$$

After multiplying both sides of (3.5) by $-c$, we have

$$-c\|u(t,\cdot)\|^p_{L_2^m} \leq -\frac{c}{b}V(u(t,\cdot)). \tag{3.6}$$

From (3.3) and (3.6) it follows that

$$D^+\left[V(u(t,\cdot))\right] \leq -\frac{c}{b}V(u(t,\cdot)). \tag{3.7}$$

To use the comparison principle, consider the ODE

$$\frac{d}{dt}\left[\phi(t)\right] = -\frac{c}{b}\phi(t), \quad \phi(0) = V(u(0,\cdot)), \tag{3.8}$$

where $t \in (0,\infty)$ and function $\phi : [0,\infty) \to \mathbb{R}$ is continuous. Solution for (3.8) is

$$\phi(t) = V(u(0,\cdot))\exp\left\{-\frac{c}{b}t\right\}$$

for all $t \geq 0$. Applying Lemma 1 for (3.7) and (3.8) results in

$$V(u(t,\cdot)) \leq V(u(0,\cdot))\exp\left\{-\frac{c}{b}t\right\} \tag{3.9}$$

for all $t \geq 0$. Substituting $t = 0$ in the second inequality of (3.4) implies

$$V(u(0,\cdot)) \leq b\|u(0,\cdot)\|^p_{L_2^m}. \tag{3.10}$$

Combining the first inequality of (3.4) with (3.10) and (3.9) gives

$$a\|u(t,\cdot)\|^l_{L_2^m} \leq V(u(t,\cdot)) \leq V(u(0,\cdot))\exp\left\{-\frac{c}{b}t\right\} \leq b\|u(0,\cdot)\|^p_{L_2^m}\exp\left\{-\frac{c}{b}t\right\}. \tag{3.11}$$

Dividing (3.11) by a and taking the l^{th} root results in

$$\|u(t,\cdot)\|_{L_2^m} \leq \sqrt[l]{\frac{b}{a}}\|u(0,\cdot)\|^{p/l}_{L_2^m}\exp\left\{-\frac{c}{lb}t\right\} \text{ for all } t \geq 0.$$

$\square$

Chapter 4

COUPLED LINEAR PDES

Recall from Dullerud and Paganini (2013) how LMIs can be used in stability analysis for linear ODEs.

4.1 Quadratic Lyapunov Functions

Theorem 3. *Given a system*

$$\frac{d}{dt}[x(t)] = Ax(t) \tag{4.1}$$

where $t > 0$*,* $x : (0, \infty) \to \mathbb{R}^n$ *and* $A \in \mathbb{R}^{n \times n}$*, if*

$$\exists\ P > 0 \text{ such that } A^T P + PA < 0 \tag{4.2}$$

then (4.1) is exponentially stable.

Proof. If (4.2) holds, then $V(z) = z^T P z$ is a Lyapunov function for (4.1) since $A^T P + PA < 0$ and for all $t > 0$,

$$\begin{aligned}\frac{d}{dt}[V(x(t))] &= \frac{d}{dt}[x(t)^T]Px(t) + x(t)^T P \frac{d}{dt}[x(t)] = x(t)^T A^T P x(t) + x(t)^T PAx(t) \\ &= x(t)^T (A^T P + PA)x(t) < 0.\end{aligned}$$

□

The natural question is if one can parameterize a set of positive and negative operators on functional spaces such as $L_2^m(\Omega)$ with some $\Omega \subseteq \mathbb{R}^n$. The following theorem uses positive matrices to parameterize a set of positive operators on $L_2^m(a, b)$ of the form

$$(\mathcal{P}w)(x) := M(x)w(x) + \int_a^b N(x, y)w(y)\, dy$$

for all $x \in (a, b)$ with any $a, b \in \mathbb{R}$.

Theorem 4. *Given any positive semi-definite matrix* $P \in \mathbb{S}^{\frac{m}{2}(d+1)(d+4)}$ *one can partition it as*

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \tag{4.3}$$

such that $P_{11} \in \mathbb{S}^{m(d+1)}$. *Define*

$$Z_1(x) := Z_d(x) \otimes I_m \quad \text{and} \quad Z_2(x,y) := Z_d(x,y) \otimes I_m \tag{4.4}$$

where $x, y \in (a,b)$, Z_d *is a vector of monomials up to degree* d *and* $\otimes$ *is the Kronecker product. If for some* $\epsilon > 0$

$$M(x) := Z_1(x)^T P_{11} Z_1(x) + \epsilon I_m, \tag{4.5}$$

$$N(x,y) := Z_1(x)^T P_{12} Z_2(x,y) + Z_2(y,x)^T P_{21} Z_1(y) + \int_a^b Z_2(z,x)^T P_{22} Z_2(z,y)\, dz, \tag{4.6}$$

then functional $V : L_2^m(a,b) \to \mathbb{R}$, *defined as*

$$V(w) := \int_a^b w(x)^T M(x) w(x)\, dx + \int_a^b w(x)^T \int_a^b N(x,y) w(y)\, dy dx, \tag{4.7}$$

satisfies

$$V(w) \geq \epsilon \|w\|_{L_2^m} \quad \text{for all} \quad w \in L_2^m(a,b). \tag{4.8}$$

Proof. The idea of the proof is to show that V from (4.7), satisfies the following equation.

$$V(w) = \int_a^b (\mathcal{Z}w)(x)^T P (\mathcal{Z}w)(x)\, dx + \epsilon \int_a^b w(x)^T w(x)\, dx, \tag{4.9}$$

where for all $x \in (a,b)$,

$$(\mathcal{Z}w)(x) := \begin{bmatrix} Z_1(x) w(x) \\ \int_a^b Z_2(x,y) w(y) dy \end{bmatrix}. \tag{4.10}$$

Since $P \geq 0$, then it is straightforward to show (4.8).

Consider the first integral of the right hand side in (4.9), substitute for $\mathcal{Z}$ from (4.10) and use the partition (4.3) as follows.

$$\begin{aligned}\int_a^b (\mathcal{Z}w)(x)^T P(\mathcal{Z}w)(x)\,dx = &\int_a^b w(x)^T Z_1(x)^T P_{11} Z_1(x) w(x)\,dx \\ &+ \int_a^b w(x)^T Z_1(x)^T P_{12} \int_a^b Z_2(x,y) w(y)\,dydx \\ &+ \int_a^b \int_a^b w(y)^T Z_2(x,y)^T\,dy\,P_{21} Z_1(x) w(x) dx \\ &+ \int_a^b \int_a^b w(y)^T Z_2(x,y)^T\,dy\,P_{22} \int_a^b Z_2(x,z) w(z)\,dzdx.\end{aligned} \tag{4.11}$$

Changing the order of integration in the 3rd integral of the right hand side of (4.24) and then switching between the integration variables x and y results in

$$\int_a^b \int_a^b w(y)^T Z_2(x,y)^T\,dy\,P_{21} Z_1(x) w(x) dx = \int_a^b w(x)^T \int_a^b Z_2(y,x)^T P_{21} Z_1(y) w(y)\,dydx. \tag{4.12}$$

Changing two times the order of integration in the 4th integral of the right hand side of (4.24) and then switching first between the integration variables x and z, and then between x and y results in

$$\begin{aligned}&\int_a^b \int_a^b w(y)^T Z_2(x,y)^T\,dy\,P_{22} \int_a^b Z_2(x,z) w(z)\,dz\,dx \\ &= \int_a^b \int_a^b \int_a^b w(y)^T Z_2(x,y)^T P_{22} Z_2(x,z) w(z) dxdzdy \\ &= \int_a^b \int_a^b \int_a^b w(y)^T Z_2(z,y)^T P_{22} Z_2(z,x) w(x) dzdxdy \\ &= \int_a^b \int_a^b \int_a^b w(x)^T Z_2(z,x)^T P_{22} Z_2(z,y) w(y) dzdydx \\ &= \int_a^b w(x)^T \int_a^b \int_a^b Z_2(z,x)^T P_{22} Z_2(z,y) dz w(y) dydx.\end{aligned} \tag{4.13}$$

Using (4.24)-(4.26) one can write

$$\int_a^b (\mathcal{Z}w)(x)^T P(\mathcal{Z}w)(x)\,dx = \int_a^b w(x)^T Z_1(x)^T P_{11} Z_1(x) w(x)\,dx + \int_a^b w(x)^T \int_a^b \Bigg(Z_1(x)P_{12}Z_2(x,y) + Z_2(y,x)P_{21}Z_1(y) + \int_a^b Z_2(z,x)^T P_{22} Z_2(z,y) dz \Bigg) w(y)\,dy\,dx. \tag{4.14}$$

From (4.5), (4.6) and (4.27) it follows that

$$\int_a^b (\mathcal{Z}w)(x)^T P(\mathcal{Z}w)(x)\,dx = \int_a^b w(x)^T M(x) w(x)\,dx - \epsilon \int_a^b w(x)^T w(x)\,dx + \int_a^b w(x)^T \int_a^b N(x,y) w(y)\,dydx. \tag{4.15}$$

Adding $\epsilon \int_a^b w(x)^T w(x)\,dx$ to the both sides of (4.29) and using (4.7) results in (4.9), which concludes the proof. □

4.2 Extending the Set of Lyapunov Candidates

Adding an extra term in (4.9) as follows allows to parameterize a larger set of Lyapunov candidates.

$$V(w) = \int_a^b (\mathcal{Z}w)(x)^T P(\mathcal{Z}w)(x)\,dx + \epsilon \int_a^b w(x)^T w(x)\,dx + \int_a^b g(x)(\mathcal{Z}w)(x)^T Q(\mathcal{Z}w)(x)\,dx, \tag{4.16}$$

where $g : [a,b] \to \mathbb{R}$ is continuous and positive and $Q \geq 0$. In this work we used

$$g(x) := (x-a)(b-x) \tag{4.17}$$

for all $x \in [a,b]$. Other choices for g are possible. For more information see Positivstellensatz in Stengle (1974).

Based on (4.16) Theorem 4 can be modified as follows.

Theorem 5. *Given any positive semi-definite matrices* $P, Q \in \mathbb{S}^{\frac{m}{2}(d+1)(d+4)}$ *one can partition them as*

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \text{ and } Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}, \tag{4.18}$$

such that $P_{11}, Q_{11} \in \mathbb{S}^{m(d+1)}$. *If for some* $\epsilon > 0$

$$M(x) := Z_1(x)^T \left(P_{11} + g(x)Q_{11}\right) Z_1(x) + \epsilon I_m, \tag{4.19}$$

$$\begin{aligned} N(x,y) := {} & Z_1(x)^T \left(P_{12} + g(x)Q_{12}\right) Z_2(x,y) + Z_2(y,x)^T \left(P_{21} + g(y)Q_{21}\right) Z_1(y) \\ & + \int_a^b Z_2(z,x)^T \left(P_{22} + g(z)Q_{22}\right) Z_2(z,y)\, dz, \end{aligned} \tag{4.20}$$

where as before

$$Z_1(x) := Z_d(x) \otimes I_m \quad \text{and} \quad Z_2(x,y) := Z_d(x,y) \otimes I_m$$

and some positive and continuous function g, *then functional* $V : L_2^m(a,b) \to \mathbb{R}$, *defined as*

$$V(w) := \int_a^b w(x)^T M(x) w(x)\, dx + \int_a^b w(x)^T \int_a^b N(x,y) w(y)\, dy dx, \tag{4.21}$$

satisfies $V(w) \geq \epsilon \|w\|_{L_2^m}$ *for all* $w \in L_2^m(a,b)$.

Proof. The idea of the proof is to show that V from (4.21), satisfies the following equation.

$$\begin{aligned} V(w) = \epsilon \int_a^b w(x)^T w(x)\, dx + \int_a^b (\mathcal{Z}w)(x)^T P (\mathcal{Z}w)(x)\, dx \\ + \int_a^b g(x) (\mathcal{Z}w)(x)^T Q (\mathcal{Z}w)(x)\, dx, \end{aligned} \tag{4.22}$$

where

$$(\mathcal{Z}w)(x) := \begin{bmatrix} Z_1(x) w(x) \\ \int_a^b Z_2(x,y) w(y) dy \end{bmatrix}. \tag{4.23}$$

Since $P, Q \geq 0$ and $g(x) \geq 0$ for all $x \in [a,b]$, then the right hand side of (4.22) is positive.

Consider 3rd integral of (4.22) using (4.23), since for the 2nd integral steps are almost the same, except it does not have the multiplier function g.

$$
\begin{aligned}
\int_a^b g(x)(\mathcal{Z}w)(x)^T Q(\mathcal{Z}w)(x)\,dx = &\int_a^b g(x)w(x)^T Z_1(x)^T Q_{11} Z_1(x)w(x)\,dx \\
&+ \int_a^b g(x)w(x)^T Z_1(x)^T Q_{12} \int_a^b Z_2(x,y)w(y)\,dydx \\
&+ \int_a^b g(x) \int_a^b w(y)^T Z_2(x,y)^T\,dy\,Q_{21} Z_1(x)w(x)dx \\
&+ \int_a^b g(x) \int_a^b w(y)^T Z_2(x,y)^T\,dy\,Q_{22} \int_a^b Z_2(x,z)w(z)\,dz\,dx.
\end{aligned}
\tag{4.24}
$$

Changing the order of integration in the 3rd integral of the right hand side of (4.24) and then switching between the integration variables x and y results in

$$
\begin{aligned}
&\int_a^b g(x) \int_a^b w(y)^T Z_2(x,y)^T\,dy\,Q_{21} Z_1(x)w(x)dx \\
&\qquad = \int_a^b w(x)^T \int_a^b g(y) Z_2(y,x)^T Q_{21} Z_1(y)w(y)\,dydx.
\end{aligned}
\tag{4.25}
$$

Changing two times the order of integration in the 4th integral of the right hand side of (4.24) and then switching first between the integration variables x and z, and then between x and y results in

$$
\begin{aligned}
\int_a^b g(x)\int_a^b w(y)^T Z_2(x,y)^T\,dy\,Q_{22}\int_a^b Z_2(x,z)w(z)\,dz\,dx& \\
&= \int_a^b\int_a^b\int_a^b g(x)w(y)^T Z_2(x,y)^T Q_{22}Z_2(x,z)w(z)dxdzdy \\
&= \int_a^b\int_a^b\int_a^b g(z)w(y)^T Z_2(z,y)^T Q_{22}Z_2(z,x)w(x)dzdxdy \\
&= \int_a^b\int_a^b\int_a^b g(z)w(x)^T Z_2(z,x)^T Q_{22}Z_2(z,y)w(y)dzdydx \\
&= \int_a^b w(x)^T\int_a^b\int_a^b Z_2(z,x)^T g(z)Q_{22}Z_2(z,y)dzw(y)dydx.
\end{aligned}
\tag{4.26}
$$

Using (4.24)-(4.26) one can write

$$
\begin{aligned}
\int_a^b g(x)(\mathcal{Z}w)(x)^T Q(\mathcal{Z}w)(x)\,dx = \int_a^b w(x)^T Z_1(x)^T g(x)Q_{11}Z_1(x)w(x)\,dx& \\
+\int_a^b w(x)^T\int_a^b \Bigg(Z_1(x)g(x)Q_{12}Z_2(x,y)& \\
+ Z_2(y,x)g(y)Q_{21}Z_1(y)& \\
+\int_a^b Z_2(z,x)^T g(z)Q_{22}Z_2(z,y)dz\Bigg)w(y)\,dy\,dx.&
\end{aligned}
\tag{4.27}
$$

Following the same idea as in (4.24)-(4.27) it is possible to achieve

$$
\begin{aligned}
\int_a^b (\mathcal{Z}w)(x)^T P(\mathcal{Z}w)(x)\,dx = \int_a^b w(x)^T Z_1(x)^T P_{11}Z_1(x)w(x)\,dx& \\
+\int_a^b w(x)^T\int_a^b \Bigg(Z_1(x)P_{12}Z_2(x,y) + Z_2(y,x)P_{21}Z_1(y)& \\
+\int_a^b Z_2(z,x)^T P_{22}Z_2(z,y)dz\Bigg)w(y)\,dy\,dx.&
\end{aligned}
\tag{4.28}
$$

Using (4.19), (4.20), (4.27) and (4.28) yields

$$\int_a^b (\mathcal{Z}w)(x)^T P(\mathcal{Z}w)(x)\,dx + \int_a^b g(x)(\mathcal{Z}w)(x)^T Q(\mathcal{Z}w)(x)\,dx$$
$$= \int_a^b w(x)^T M(x) w(x)\,dx - \epsilon \int_a^b w(x)^T w(x)\,dx + \int_a^b w(x)^T \int_a^b N(x,y)w(y)\,dydx. \tag{4.29}$$

Adding $\epsilon \int_a^b w(x)^T w(x)\,dx$ to the both sides of (4.29) and using (4.21)results in (4.9), which concludes the proof. $\square$

For simplicity, define a set of polynomials (M, N) as follows.

$$\Sigma_+^{m,d,\epsilon} := \{(M,N) : \exists\ P, Q \geq 0 \text{ and } (4.19), (4.20) \text{ hold}\}. \tag{4.30}$$

Similarly, define a different set of polynomials for some $\epsilon < 0$ that parameterize functionals of the form (4.21) such that $V(w) \leq \epsilon \|w\|_{L_2^m}$ for all $w \in L_2^m(a,b)$. Denote

$$\Sigma_-^{m,d,\epsilon} := \{(M,N) : (-M,-N) \in \Sigma_+^{m,d,-\epsilon}\}. \tag{4.31}$$

4.3 Quadratic From of the Time Derivative of Lyapunov Function

In this section a quadratic form of the time derivative of Lyapunov function is presented. First recall the PDE of interest. For all $t \in (0,\infty)$ and $x \in (a,b) \subset \mathbb{R}$, $u : [0,\infty) \times [a,b] \to \mathbb{R}^m$ satisfies

$$u_t(t,x) = A(x)u_{xx}(t,x) + B(x)u_x(t,x) + C(x)u(t,x), \tag{4.32}$$

where A, B, C are some given polynomial matrices.

Substituting $u(t,\cdot)$ for w in (4.7) and differentiating with respect to t results in

$$\begin{aligned}\frac{d}{dt}[V(u(t,\cdot))] &= \frac{d}{dt}\left[\int_a^b u(t,x)^T M(x)u(t,x)\,dx + \int_a^b u(t,x)^T \int_a^b N(x,y)u(t,y)\,dydx\right] \\ &= \int_a^b \big(u_t(t,x)^T M(x)u(t,x) + u(t,x)^T M(x)u_t(t,x)\big)dx \\ &+ \int_a^b \left(u_t(t,x)^T \int_a^b N(x,y)u(t,y)\,dy + u(t,x)^T \int_a^b N(x,y)u_t(t,y)\,dy\right)dx.\end{aligned} \tag{4.33}$$

Now substituting for u_t from (4.32) into (4.33) yields

$$\begin{aligned}\frac{d}{dt}[V(u(t,\cdot))] = \int_a^b \Bigg(&\Big(A(x)u_{xx}(t,x) + B(x)u_x(t,x) + C(x)u(t,x)\Big)^T M(x)u(t,x) \\ &+ u(t,x)^T M(x)\Big(A(x)u_{xx}(t,x) + B(x)u_x(t,x) + C(x)u(t,x)\Big)\Bigg)\,dx \\ &+ \int_a^b \Bigg(\Big(A(x)u_{xx}(t,x) + B(x)u_x(t,x) \\ &\quad + C(x)u(t,x)\Big)^T \int_a^b N(x,y)u(t,y)\,dy \\ &\quad\quad + u(t,x)^T \int_a^b N(x,y)\Big(A(y)u_{yy}(t,y) \\ &\quad\quad\quad + B(y)u_y(t,y) + C(y)u(t,y)\Big)\,dy\Bigg)dx.\end{aligned} \tag{4.34}$$

If one defines

$$K(x) := \begin{bmatrix} C(x)^T M(x) + M(x)C(x)(x) & M(x)B(x) & M(x)A(x) \\ B(x)^T M(x) & 0 & 0 \\ A(x)^T M(x) & 0 & 0 \end{bmatrix},$$

$$L(x,y) := \begin{bmatrix} C(x)^T N(x,y) + N(x,y)C(y) & N(x,y)B(y) & N(x,y)A(y) \\ B(x)^T N(x,y) & 0 & 0 \\ A(x)^T N(x,y) & 0 & 0 \end{bmatrix},$$

$$q(t,x) := \begin{bmatrix} u(t,x) \\ u_x(t,x) \\ u_{xx}(t,x) \end{bmatrix},$$

then (4.34) can be written as

$$\frac{d}{dt}\left[V(u(t,\cdot))\right] = \int_a^b q(t,x)^T K(x) q(t,x)\, dx + \int_a^b q(t,x)^T L(x,y) q(t,y)\, dydx. \quad (4.35)$$

Equation (4.35) represents quadratic form of the time derivative of Lyapunov candidate. If $(K, L) \in \Sigma_-^{3m,d,0}$ then for every $t > 0$

$$\frac{d}{dt}\left[V(u(t,\cdot))\right] \leq 0$$

and, therefore, V is a Lyapunov function, thus PDE (4.32) is stable. If $(K, L) \in \Sigma_-^{3m,d,\epsilon}$ with some $\epsilon < 0$ then (4.32) is exponentially stable.

Notice, that condition $(K, L) \in \Sigma_-^{3m,d,0}$ is conservative. The reason is that the elements in U are not independent, i.e. the second and third elements are partial derivatives of the first one. Therefore, for the PDE (4.32) to be stable, it is enough to check if (4.35) is negative on a subspace of $L_2^{3m}(a,b)$, which is

$$\Lambda = \left\{ \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \in L_2^{3m}(a,b) : D \begin{bmatrix} w_1(a) \\ w_1(b) \\ w_2(a) \\ w_2(b) \end{bmatrix} = 0, \begin{array}{c} w_2 = w_1', \\ w_3 = w_1'' \end{array} \right\} \quad (4.36)$$

Notice, that Λ depends on D that represents the boundary conditions as before in (1.3).

4.4 Spacing Operators

Results of the following theorem are used to parameterize functions which are negative on Λ, but not necessarily on the whole space $L_2^{3m}(a,b)$.

Theorem 6. *Let X be a closed subspace of some Hilbert space Y. Then $\langle u, \mathcal{R}u\rangle_Y \le 0$ for all $u \in X$ if and only if there exist $\mathcal{M}$ and $\mathcal{T}$ such that $\mathcal{R} = \mathcal{M} + \mathcal{T}$ and $\langle w, \mathcal{M}w\rangle_Y \le 0$ for all $w \in Y$ and $\langle u, \mathcal{T}u\rangle_Y = 0$ for all $u \in X$.*

Proof. For ($\Rightarrow$), suppose that $\langle u, \mathcal{R}u\rangle_Y \le 0$ for all $u \in X$. Since X is a closed subspace of a Hilbert space Y, there exists a projection operator such that $\mathcal{P} = \mathcal{P}^* = \mathcal{P}\mathcal{P}$ and $\mathcal{P}w \in X$ for all $w \in Y$. Let $\mathcal{M} = \mathcal{P}\mathcal{R}\mathcal{P}$ and $\mathcal{T} = \mathcal{M} - \mathcal{R}$. Then for all $w \in Y$,

$$\langle w, \mathcal{M}w\rangle_Y = \langle w, \mathcal{P}\mathcal{R}\mathcal{P}w\rangle_Y = \langle \mathcal{P}w, \mathcal{R}\mathcal{P}w\rangle_Y \le 0$$

since $\mathcal{P}w \in X$. Furthermore, for all $u \in X$

$$\begin{aligned}\langle u, \mathcal{T}u\rangle_Y &= \langle u, \mathcal{P}\mathcal{R}\mathcal{P}u\rangle_Y - \langle u, \mathcal{R}u\rangle_Y = \langle \mathcal{P}u, \mathcal{R}\mathcal{P}u\rangle_Y - \langle u, \mathcal{R}u\rangle_Y \\ &= \langle u, \mathcal{R}u\rangle_Y - \langle u, \mathcal{R}u\rangle_Y = 0.\end{aligned}$$

For ($\Leftarrow$), assume that there exist $\mathcal{M}$ and $\mathcal{T}$ such that $\mathcal{R} = \mathcal{M}+\mathcal{T}$ and $\langle w, \mathcal{M}w\rangle_Y \le 0$ for all $w \in Y$ and $\langle u, \mathcal{T}u\rangle_Y = 0$ for all $u \in X$. Then for all $u \in X$,

$$\langle u, \mathcal{R}u\rangle_Y = \langle u, (\mathcal{M}+\mathcal{T})u\rangle_Y = \langle u, \mathcal{M}u\rangle_Y + \langle u, \mathcal{T}u\rangle_Y = \langle u, \mathcal{M}u\rangle_Y \le 0.$$

□

As was shown previously, $\Sigma_-^{3n,d,\epsilon}$ parameterizes a subset of $\mathcal{M}$. Next step is to parameterize a subset of operators $\mathcal{T}$ - the so-called "spacing operators" using polynomial spacing functions. Therefore the sum of $\mathcal{M}$ and $\mathcal{T}$ yield an operator $\mathcal{R}$ which is negative on Λ, but not necessarily on $L_2^{3m}(a,b)$ space.

4.5 Parametrization of Spacing Operators by Polynomials

The following lemmas define the structure of polynomial matrices T and R such that for all $\lambda \in \Lambda$

$$\int_a^b \lambda(x)^T T(x)\lambda(x)dx + \int_a^b \lambda(x)^T \int_a^b R(x,y)\lambda(y)dydx = 0,$$

where as before

$$\Lambda = \left\{ \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \in L_2^{3m}(a,b) : D \begin{bmatrix} w_1(a) \\ w_1(b) \\ w_2(a) \\ w_2(b) \end{bmatrix} = 0, \quad \begin{matrix} w_2 = w_1', \\ w_3 = w_1'' \end{matrix} \right\}. \tag{4.37}$$

Lemma 2. *Let $P_1, P_2, P_3, P_4 : [a,b] \to \mathbb{R}^{m\times m}$ be polynomials and $w, w', w'' \in L_2^m(a,b)$. If*

$$T(x) = \begin{bmatrix} P_1'(x) & P_1(x) + P_2'(x) & P_2(x) \\ P_1(x) + P_3'(x) & P_2(x) + P_3(x) + P_4'(x) & P_4(x) \\ P_3(x) & P_4(x) & 0 \end{bmatrix} \tag{4.38}$$

then

$$\int_a^b \lambda(x)^T T(x)\lambda(x)\,dx = q^T \Pi_1 q, \tag{4.39}$$

where

$$\lambda(x) := \begin{bmatrix} w(x) \\ w'(x) \\ w''(x) \end{bmatrix}, \; q := \begin{bmatrix} w(a) \\ w(b) \\ w'(a) \\ w'(b) \end{bmatrix}, \; \Pi_1 := \begin{bmatrix} -P_1(a) & 0 & -P_2(a) & 0 \\ 0 & P_1(b) & 0 & P_2(a) \\ -P_3(a) & 0 & -P_4(a) & 0 \\ 0 & P_3(b) & 0 & P_4(b) \end{bmatrix}. \tag{4.40}$$

Proof. Using the fundamental theorem of calculus it is true that

$$\int_a^b \frac{d}{dx}\left(\begin{bmatrix} w(x)^T \\ w'(x)^T \end{bmatrix}^T \begin{bmatrix} P_1(x) & P_2(x) \\ P_3(x) & P_4(x) \end{bmatrix} \begin{bmatrix} w(x) \\ w'(x) \end{bmatrix}\right) dx$$

$$= \begin{bmatrix} w(b)^T \\ w'(b)^T \end{bmatrix}^T \begin{bmatrix} P_1(b) & P_2(b) \\ P_3(b) & P_4(b) \end{bmatrix} \begin{bmatrix} w(b) \\ w'(b) \end{bmatrix}$$

$$- \begin{bmatrix} w(a)^T \\ w'(a)^T \end{bmatrix}^T \begin{bmatrix} P_1(a) & P_2(a) \\ P_3(a) & P_4(a) \end{bmatrix} \begin{bmatrix} w(a) \\ w'(a) \end{bmatrix}$$

$$= \begin{bmatrix} w(a)^T \\ w(b)^T \\ w'(a)^T \\ w'(b)^T \end{bmatrix}^T \begin{bmatrix} -P_1(a) & 0 & -P_2(a) & 0 \\ 0 & P_1(b) & 0 & P_2(a) \\ -P_3(a) & 0 & -P_4(a) & 0 \\ 0 & P_3(b) & 0 & P_4(b) \end{bmatrix} \begin{bmatrix} w(a) \\ w(b) \\ w'(a) \\ w'(b) \end{bmatrix}$$

$$= q^T \Pi_1 q. \tag{4.41}$$

From the other side, using chain rule it can be seen that

$$\frac{d}{dx}\left(\begin{bmatrix} w(x)^T \\ w'(x)^T \end{bmatrix}^T \begin{bmatrix} P_1(x) & P_2(x) \\ P_3(x) & P_4(x) \end{bmatrix} \begin{bmatrix} w(x) \\ w'(x) \end{bmatrix}\right)$$

$$= \begin{bmatrix} w(x)^T \\ w'(x)^T \\ w''(x)^T \end{bmatrix}^T \begin{bmatrix} P_1'(x) & P_1(x) + P_2'(x) & P_2(x) \\ P_1(x) + P_3'(x) & P_2(x) + P_3(x) + P_4'(x) & P_4(x) \\ P_3(x) & P_4(x) & 0 \end{bmatrix} \begin{bmatrix} w(x) \\ w'(x) \\ w''(x) \end{bmatrix}$$

$$= \lambda(x)^T T(x) \lambda(x). \tag{4.42}$$

Combining (4.41) and (4.42) results in (4.39). ☐

Notice that $Dq = 0$ and, therefore,

$$\begin{aligned}
q^T\Pi_1 q &= q^T(I_{4m} - D + D)^T\Pi_1(I_{4m} - D + D)q \\
&= q^T(I_{4m} - D + D)^T\Pi_1(I_{4m} - D)q = q^T(I_{4m} - D)^T\Pi_1(I_{4m} - D)q.
\end{aligned}$$

Using Lemma (2) one can define the following set.

$$\Xi_1^D := \{T \text{ as defined in (4.38)} : (I_{4m} - D)^T\Pi_1(I_{4m} - D) = 0,\ \Pi_1 \text{ as defined in (4.40)}\}$$

Thus, for any $T \in \Xi_1^D$ and any $\lambda \in \Lambda$ it is true that

$$\int_a^b \lambda(x)^T T(x)\lambda(x)dx = 0.$$

Lemma 3. *Let $Q_1, Q_2, Q_3, Q_4 : [a, b] \times [a, b] \to \mathbb{R}^{m\times m}$ be polynomials and $w, w', w'' \in L_2^m(a, b)$. If*

$$R_1(x,y) := \begin{bmatrix} Q_{1,xy}(x,y) & Q_{3,xy}(x,y) + Q_{1,x}(x,y) & Q_{3,x}(x,y) \\ Q_{2,xy}(x,y) + Q_{1,y}(x,y) & R_{22}(x,y) & Q_{4,x}(x,y) + Q_3(x,y) \\ Q_{2,y}(x,y) & Q_{4,y}(x,y) + Q_2(x,y) & Q_4(x,y) \end{bmatrix}$$

$$R_{22}(x,y) := Q_{4,xy}(x,y) + Q_{2,x}(x,y) + Q_{3,y}(x,y), \tag{4.43}$$

then

$$\int_a^b\int_a^b \lambda(x)^T R_1(x,y)\lambda(y)dxdy = q^T\Theta_1 q,$$

where

$$\lambda(x) := \begin{bmatrix} w(x) \\ w'(x) \\ w''(x) \end{bmatrix}, \; q := \begin{bmatrix} w(a) \\ w(b) \\ w'(a) \\ w'(b) \end{bmatrix},$$

$$\Theta_1 := \begin{bmatrix} Q_1(a,a) & -Q_1(a,b) & Q_3(a,a) & -Q_3(a,b) \\ -Q_1(b,a) & Q_1(b,b) & -Q_3(b,a) & Q_3(b,b) \\ Q_2(a,a) & -Q_2(a,b) & Q_4(a,a) & -Q_4(a,b) \\ -Q_2(b,a) & Q_2(b,b) & -Q_4(b,a) & Q_4(b,b) \end{bmatrix}. \tag{4.44}$$

Proof. Applying the fundamental theorem of calculus twice to

$$\begin{aligned}
&\int_a^b \int_a^b \frac{\partial^2}{\partial x \partial y}\left(\begin{bmatrix} w(x)^T \\ w'(x)^T \end{bmatrix}^T \begin{bmatrix} Q_1(x,y) & Q_3(x,y) \\ Q_2(x,y) & Q_4(x,y) \end{bmatrix} \begin{bmatrix} w(y) \\ w'(y) \end{bmatrix}\right) dxdy \\
&\quad = \int_a^b \frac{\partial}{\partial y}\Bigg(\begin{bmatrix} w(b)^T \\ w'(b)^T \end{bmatrix}^T \begin{bmatrix} Q_1(b,y) & Q_3(b,y) \\ Q_2(b,y) & Q_4(b,y) \end{bmatrix} \begin{bmatrix} w(y) \\ w'(y) \end{bmatrix} \\
&\qquad\qquad - \begin{bmatrix} w(a)^T \\ w'(a)^T \end{bmatrix}^T \begin{bmatrix} Q_1(a,y) & Q_3(a,y) \\ Q_2(a,y) & Q_4(a,y) \end{bmatrix} \begin{bmatrix} w(y) \\ w'(y) \end{bmatrix}\Bigg) dy \\
&\quad = \begin{bmatrix} w(b)^T \\ w'(b)^T \end{bmatrix}^T \begin{bmatrix} Q_1(b,b) & Q_3(b,b) \\ Q_2(b,b) & Q_4(b,b) \end{bmatrix} \begin{bmatrix} w(b) \\ w'(b) \end{bmatrix} \\
&\qquad - \begin{bmatrix} w(a)^T \\ w'(a)^T \end{bmatrix}^T \begin{bmatrix} Q_1(a,b) & Q_3(a,b) \\ Q_2(a,b) & Q_4(a,b) \end{bmatrix} \begin{bmatrix} w(b) \\ w'(b) \end{bmatrix} \\
&\qquad - \begin{bmatrix} w(b)^T \\ w'(b)^T \end{bmatrix}^T \begin{bmatrix} Q_1(b,a) & Q_3(b,a) \\ Q_2(b,a) & Q_4(b,a) \end{bmatrix} \begin{bmatrix} w(a) \\ w'(a) \end{bmatrix} \\
&\qquad + \begin{bmatrix} w(a)^T \\ w'(a)^T \end{bmatrix}^T \begin{bmatrix} Q_1(a,a) & Q_3(a,a) \\ Q_2(a,a) & Q_4(a,a) \end{bmatrix} \begin{bmatrix} w(a) \\ w'(a) \end{bmatrix} \\
&\quad = \begin{bmatrix} w(a)^T \\ w(b)^T \\ w'(a)^T \\ w'(b)^T \end{bmatrix}^T \begin{bmatrix} Q_1(a,a) & -Q_1(a,b) & Q_3(a,a) & -Q_3(a,b) \\ -Q_1(b,a) & Q_1(b,b) & -Q_3(b,a) & Q_3(b,b) \\ Q_2(a,a) & -Q_2(a,b) & Q_4(a,a) & -Q_4(a,b) \\ -Q_2(b,a) & Q_2(b,b) & -Q_4(b,a) & Q_4(b,b) \end{bmatrix} \begin{bmatrix} w(a) \\ w(b) \\ w'(a) \\ w'(b) \end{bmatrix} \\
&\quad = q^T \Theta_1 q.
\end{aligned} \tag{4.45}$$

From the other side, using the chain rule, one can get

$$\frac{\partial^2}{\partial x \partial y}\left(\begin{bmatrix} w(x)^T \\ w'(x)^T \end{bmatrix}^T \begin{bmatrix} Q_1(x,y) & Q_3(x,y) \\ Q_2(x,y) & Q_4(x,y) \end{bmatrix} \begin{bmatrix} w(y) \\ w'(y) \end{bmatrix}\right)$$

$$= \begin{bmatrix} w(x)^T \\ w'(x)^T \\ w''(x) \end{bmatrix}^T R_1(x,y) \begin{bmatrix} w(x) \\ w'(x) \\ w''(x) \end{bmatrix} = \lambda(x)^T R_1(x,y)\lambda(y), \tag{4.46}$$

where R_1 is defined in (4.43). Then combining (4.45) and (4.46) finishes the proof. $\square$

As before, notice that

$$\begin{aligned} q^T\Theta_1 q &= q^T(I_{4m} - D + D)^T\Theta_1(I_{4m} - D + D)q \\ &= q^T(I_{4m} - D + D)^T\Theta_1(I_{4m} - D)q = q^T(I_{4m} - D)^T\Theta_1(I_{4m} - D)q. \end{aligned}$$

Similarly as for Ξ_1, using Lemma (3) one can define a set

$\Xi_2^D := \{R_1$ as defined in (4.43) : $(I_{4n} - D)^T\Theta_1(I_{4n} - D) = 0,\ \Theta_1$ as defined in (4.44)$\}$.

Thus, for any $R_1 \in \Xi_2^D$ and any $\lambda \in \Lambda$,

$$\int_a^b \int_a^b \lambda(x)^T R_1(x,y)\lambda(y)dxdy = 0.$$

Lemma 4. *Let $Q_5, Q_6 : [a,b]\times[a,b] \to \mathbb{R}^{m\times m}$ be polynomials and $w, w', w'' \in L_2^m(a,b)$. If*

$$R_2(x,y) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ Q_{5,y}(x,y) & Q_{6,y}(x,y) + Q_5(x,y) & Q_6(x,y) \end{bmatrix} \tag{4.47}$$

then

$$\int_a^b \int_a^b \lambda(x)^T R_2(x,y)\lambda(y)dxdy = \int_a^b w''(x)^T\Theta_2(x)qdx,$$

where

$$\Theta_2(x) = \begin{bmatrix} -Q_5(x,a) & Q_5(x,b) & -Q_6(x,a) & Q_6(x,b) \end{bmatrix}. \tag{4.48}$$

Proof. Start with applying the fundamental theorem of calculus to

$$\int_a^b \int_a^b \frac{\partial}{\partial y}\left(w''(x)^T \begin{bmatrix} Q_5(x,y) & Q_6(x,y) \end{bmatrix} \begin{bmatrix} w(y) \\ w'(y) \end{bmatrix} \right) dxdy$$

$$= \int_a^b \Big(w''(x)^T \begin{bmatrix} Q_5(x,b) & Q_6(x,b) \end{bmatrix} \begin{bmatrix} w(b) \\ w'(b) \end{bmatrix}$$

$$- w''(x)^T \begin{bmatrix} Q_5(x,a) & Q_6(x,a) \end{bmatrix} \begin{bmatrix} w(a) \\ w'(a) \end{bmatrix} \Big) dx$$

$$= \int_a^b w''(x)^T \begin{bmatrix} -Q_5(x,a) & Q_5(x,b) & -Q_6(x,a) & Q_6(x,b) \end{bmatrix} \begin{bmatrix} w(a) \\ w(b) \\ w'(a) \\ w'(b) \end{bmatrix} dx. \tag{4.49}$$

Using the chain rule one can get

$$\frac{\partial}{\partial y}\left(w''(x)^T \begin{bmatrix} Q_5(x,y) & Q_6(x,y) \end{bmatrix} \begin{bmatrix} w(y) \\ w'(y) \end{bmatrix} \right) =$$

$$\begin{bmatrix} w(x)^T \\ w'(x)^T \\ w''(x)^T \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ Q_{5,y}(x,y) & Q_{6,y}(x,y) + Q_5(x,y) & Q_6(x,y) \end{bmatrix} \begin{bmatrix} w(y) \\ w'(y) \\ w''(y) \end{bmatrix}. \tag{4.50}$$

Combining (4.49) and (4.50) concludes the proof. ☐

As previously, a set can be defined using Lemma (4).

$$\Xi_3^D := \left\{ R_2 \text{ as defined in (4.47)} : \begin{array}{c} \Theta_2(x)^T(I_{4n} - D) = 0 \text{ for all } x \in (a,b), \\ \Theta_2 \text{ as defined in (4.48)} \end{array} \right\}$$

Therefore, for any $R_1 \in \Xi_3^D$ and any $\lambda \in \Lambda$,

$$\int_a^b \int_a^b \lambda(x)^T R_2(x,y)\lambda(y)dxdy = 0.$$

Lemma 5. *Let $Q_7, Q_8 : [a,b]\times[a,b] \to \mathbb{R}^{m\times m}$ be polynomials and $w, w', w'' \in L_2^m(a,b)$. If*

$$R_3(x,y) = \begin{bmatrix} 0 & 0 & Q_{7,x}(x,y) \\ 0 & 0 & Q_{8,x}(x,y) + Q_7(x,y) \\ 0 & 0 & Q_8(x,y) \end{bmatrix} \tag{4.51}$$

then

$$\int_a^b \int_a^b \lambda(x)^T R_3(x,y)\lambda(y)dxdy = \int_a^b q^T \Theta_3(y) w''(y)dy,$$

where

$$\lambda(x) := \begin{bmatrix} w(x) \\ w'(x) \\ w''(x) \end{bmatrix}, \; q := \begin{bmatrix} w(a) \\ w(b) \\ w'(a) \\ w'(b) \end{bmatrix}, \; \Theta_3(y) = \begin{bmatrix} -Q_7(a,y) \\ Q_7(b,y) \\ -Q_8(a,y) \\ Q_8(b,y) \end{bmatrix}. \tag{4.52}$$

Proof. Apply the fundamental theorem of calculus to

$$\int_a^b \int_a^b \frac{\partial}{\partial x}\left(\begin{bmatrix} w(x)^T \\ w'(x)^T \end{bmatrix}^T \begin{bmatrix} Q_7(x,y) \\ Q_8(x,y) \end{bmatrix} w''(y) \right) dxdy$$

$$= \int_a^b \Big(\begin{bmatrix} w(b)^T \\ w'(b)^T \end{bmatrix}^T \begin{bmatrix} Q_7(b,y) \\ Q_8(b,y) \end{bmatrix} w''(y) - \begin{bmatrix} w(a)^T \\ w'(a)^T \end{bmatrix}^T \begin{bmatrix} Q_7(a,y) \\ Q_8(a,y) \end{bmatrix} w''(y) \Big) dy$$

$$= \int_a^b \begin{bmatrix} w(a)^T \\ w(b)^T \\ w'(a)^T \\ w'(b)^T \end{bmatrix}^T \begin{bmatrix} -Q_7(a,y) \\ Q_7(b,y) \\ -Q_8(a,y) \\ Q_8(b,y) \end{bmatrix} w''(y)dy. \tag{4.53}$$

Using the chain rule one can get

$$\frac{\partial}{\partial x}\left(\begin{bmatrix} w(x)^T \\ w'(x)^T \end{bmatrix}^T \begin{bmatrix} Q_7(x,y) \\ Q_8(x,y) \end{bmatrix} w''(y)\right)$$
$$= \begin{bmatrix} w(x)^T \\ w'(x)^T \\ w''(x)^T \end{bmatrix}^T \begin{bmatrix} 0 & 0 & Q_{7,x}(x,y) \\ 0 & 0 & Q_{8,x}(x,y)+Q_7(x,y) \\ 0 & 0 & Q_8(x,y) \end{bmatrix} \begin{bmatrix} w(y) \\ w'(y) \\ w''(y) \end{bmatrix}. \quad (4.54)$$

Combining (4.53) and (4.54) concludes the proof. □

From Lemma (5) one can define the following set.

$$\Xi_4^D := \left\{ R_3 \text{ as defined in (4.51)} : \begin{array}{c} (I_{4n}-D)^T\Theta_3(y)=0 \text{ for all } y\in(a,b), \\ \Theta_3 \text{ as defined in (4.52)} \end{array} \right\}.$$

Thus, for any $R_3 \in \Xi_4^D$ and any $\lambda \in \Lambda$,

$$\int_a^b \int_a^b \lambda(x)^T R_3(x,y)\lambda(y)dxdy = 0.$$

Finally, combining Ξ_i results in a set of polynomials that parameterize a subset of spacing operators, i.e.

$$\Sigma_0^{m,d,D} := \{(T,R) : T \in \Xi_1^D \text{ and } R \in \sum_{i=2}^{4} \Xi_i^D\}, \quad (4.55)$$

which provides a pair of polynomials (T,R) such that for all $\lambda \in \Lambda$

$$\int_a^b \lambda(x)^T T(x)\lambda(x)dx + \int_a^b \int_a^b \lambda(x)^T R(x,y)\lambda(y)dydx = 0.$$

4.6 An LMI Condition for Stability

This section summarizes the results of the Chapter 4 in the following theorem.

Theorem 7. *Suppose that for all $t \in (0,\infty)$ and $x \in (a,b) \subset \mathbb{R}$, $u : [0,\infty) \times [a,b] \to \mathbb{R}^m$ satisfies*

$$u_t(t,x) = A(x)u_{xx}(t,x) + B(x)u_x(t,x) + C(x)u(t,x), \tag{4.56}$$

where A, B, C are some given polynomial matrices with $\gamma = \max\{deg(A), deg(B), deg(C)\}$. If there exist $d \in \mathbb{N}$, $\epsilon_1 > 0$, $\epsilon_2 < 0$,

$$(M,N) \in \Sigma_+^{m,d,\epsilon_1},\ (T,R) \in \Sigma_0^{m,2d+2+\gamma,D},\ (H,G) \in \Sigma_-^{3m,d+\gamma,\epsilon_2}$$

such that for all $x, y \in (a,b)$

$$\begin{bmatrix} C(x)^T M(x) + M(x)C(x) & M(x)B(x) & M(x)A(x) \\ B(x)^T M(x) & 0 & 0 \\ A(x)^T M(x) & 0 & 0 \end{bmatrix} = T(x) + H(x),$$

$$\begin{bmatrix} C(x)^T N(x,y) + N(x,y)C(y) & N(x,y)B(y) & N(x,y)A(y) \\ B(x)^T N(x,y) & 0 & 0 \\ A(x)^T N(x,y) & 0 & 0 \end{bmatrix} = R(x,y) + G(x,y),$$

then (4.56) is exponentially stable.

Proof. Suppose conditions of the Theorem 7 hold. Then V as defined in (4.7) satisfies (4.8). Since M and N are polynomials, they are continuous. Thus there exists $b \in \mathbb{R}$ such that

$$V(w) \leq b\|w\|_{L_2^n}.$$

According to (4.34) and (4.35) the time derivative of V satisfies

$$\frac{d}{dt}[V(u(t,\cdot))] \leq \epsilon_2\|w\|_{L_2^n}$$

and, therefore, Theorem 2 can be applied concluding the proof. □

Chapter 5

PDES WITH TWO SPATIAL VARIABLES

In this chapter the PDEs of interest have 2 spatial variables. Specifically, for all $t > 0$ and $x \in \Omega := (0,1)^2$, $u : [0,\infty) \times \Omega \to \mathbb{R}$ satisfies

$$\begin{aligned} u_t(t,x) &= a(x)u_{x_1x_1}(t,x) + b(x)u_{x_1x_2}(t,x) + c(x)u_{x_2x_2}(t,x) + d(x)u_{x_1}(t,x) \\ &\quad + e(x)u_{x_2}(t,x) + f(x)u(t,x), \end{aligned} \tag{5.1}$$

where a, b, c, d, e, f are polynomials. As before, assume that solution to (5.1) exists, is unique and depends continuously on initial conditions. Let u satisfy zero Dirichlet boundary conditions, i.e.

$$u(t,1,x_2) = 0,\ u(t,0,x_2) = 0,\ u(t,x_1,1) = 0,\ u(t,x_1,0) = 0 \tag{5.2}$$

for all $x_1, x_2 \in [0,1]$ and $t \geq 0$.

5.1 Parameterizating Lyapunov Candidates with SOS Polynomials

Define $V : L_2(\Omega) \to \mathbb{R}$ as

$$V(v) := \int_\Omega s(x)v(x)^2\,dx, \tag{5.3}$$

where s is an SOS polynomial.

5.2 Quadratic Form of the Lyapunov Time Derivative

Using $u(t,\cdot)$ for v in (5.3) and differentiating the result with respect to t gives

$$\frac{d}{dt}\left[V(u(t,\cdot))\right] = \frac{d}{dt}\left[\int_\Omega s(x)u(t,x)^2\,dx\right] = \int_\Omega 2s(x)u(t,x)u_t(t,x)\,dx. \tag{5.4}$$

Substituting for $u_t(t,x)$ from (5.1) into (5.4) implies

$$\begin{aligned}\frac{d}{dt}\left[V(u(t,\cdot))\right] = \int_\Omega 2s(x)u(t,x)\Big(&a(x)u_{x_1x_1}(t,x) + b(x)u_{x_1x_2}(t,x) + c(x)u_{x_2x_2}(t,x)\\ &+ d(x)u_{x_1}(t,x) + e(x)u_{x_2}(t,x) + f(x)u(t,x)\Big)\,dx\\ = I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t),&\end{aligned} \tag{5.5}$$

where

$$\begin{aligned}
I_1(t) &:= \int_\Omega 2s(x)u(t,x)a(x)u_{x_1x_1}(t,x)\,dx,\\
I_2(t) &:= \int_\Omega s(x)u(t,x)b(x)u_{x_2x_1}(t,x)\,dx,\\
I_3(t) &:= \int_\Omega s(x)u(t,x)b(x)u_{x_1x_2}(t,x)\,dx,\\
I_4(t) &:= \int_\Omega 2s(x)u(t,x)c(x)u_{x_2x_2}(t,x)\,dx,\\
I_5(t) &:= \int_\Omega 2s(x)u(t,x)\Big(d(x)u_{x_1}(t,x) + e(x)u_{x_2}(t,x) + f(x)u(t,x)\Big)\,dx.
\end{aligned}$$

Note that, based on section 5.2.3 of Evans (1998), it holds that

$$u_{x_1x_2}(t,x) = u_{x_2x_1}(t,x) \tag{5.6}$$

for all $x \in \Omega$. Property (5.6) was used to define I_2 and I_3. Alternatively, I_5 can be formulated as

$$I_5(t) = \int_\Omega q^T(t,x)Z_5(x)q(t,x)\,dx, \tag{5.7}$$

where for all $x \in \Omega$

$$q(t,x) := \begin{bmatrix} u(t,x)\\ u_{x_1}(t,x)\\ u_{x_2}(t,x)\end{bmatrix},\ Z_5(x) := \begin{bmatrix} 2s(x)f(x) & s(x)d(x) & s(x)e(x)\\ s(x)d(x) & 0 & 0\\ s(x)e(x) & 0 & 0\end{bmatrix}. \tag{5.8}$$

Using integration by parts and boundary conditions (5.2), I_1 can be rewritten as follows.

$$\begin{aligned}
I_1(t) &= \int_\Omega 2s(x)u(t,x)a(x)\frac{d}{dx_1}[u_{x_1}(t,x)]\,dx \\
&= 2\int_0^1 \Big(s(x)u(t,x)a(x)u_{x_1}(t,x)|_{x_1=0}^{x_1=1} \\
&\qquad - \int_0^1 u_{x_1}(t,x)\frac{d}{dx_1}[s(x)u(t,x)a(x)]\,dx_1\Big)\,dx_2 \\
&= -\int_\Omega 2u_{x_1}(t,x)\Big(u(t,x)\frac{d}{dx_1}[s(x)a(x)] + s(x)a(x)u_{x_1}(t,x)\Big)\,dx \\
&= -\int_\Omega q^T(t,x)Z_1(x)q(t,x)\,dx, \qquad (5.9)
\end{aligned}$$

where for all $x \in \Omega$

$$Z_1(x) := \begin{bmatrix} 0 & \frac{d}{dx_1}[s(x)a(x)] & 0 \\ \frac{d}{dx_1}[s(x)a(x)] & 2s(x)a(x) & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Following steps of (5.9) for I_2, I_3 and I_4, we get

$$\begin{aligned}
I_2(t) &= \int_\Omega s(x)u(t,x)b(x)\frac{d}{dx_1}[u_{x_2}(t,x)]\,dx \\
&= \int_0^1 \Big(s(x)u(t,x)b(x)u_{x_2}(t,x)|_{x_1=0}^{x_1=1} - \int_0^1 u_{x_2}(t,x)\frac{d}{dx_1}[s(x)u(t,x)b(x)]\,dx_1\Big)dx_2 \\
&= -\int_\Omega q^T(t,x)Z_2(x)q(t,x)\,dx, \\
I_3(t) &= \int_\Omega s(x)u(t,x)b(x)\frac{d}{dx_2}[u_{x_1}(t,x)]\,dx \\
&= \int_0^1 \Big(s(x)u(t,x)b(x)u_{x_1}(t,x)|_{x_2=0}^{x_2=1} - \int_0^1 u_{x_1}(t,x)\frac{d}{dx_2}[s(x)u(t,x)b(x)]dx_2\Big)\,dx_1 \\
&= -\int_\Omega q^T(t,x)Z_3(x)q(t,x)\,dx, \\
I_4(t) &= \int_\Omega 2s(x)u(t,x)c(x)\frac{d}{dx_2}[u_{x_2}(t,x)]\,dx = -\int_\Omega q^T(t,x)Z_4(x)q(t,x)\,dx, \qquad (5.10)
\end{aligned}$$

where q is defined as in (5.8) and for all $x \in \Omega$

$$Z_2(x) := \begin{bmatrix} 0 & 0 & \frac{1}{2}\frac{d}{dx_1}[s(x)b(x)] \\ 0 & 0 & \frac{1}{2}s(x)b(x) \\ \frac{1}{2}\frac{d}{dx_1}[s(x)b(x)] & \frac{1}{2}s(x)b(x) & 0 \end{bmatrix},$$

$$Z_3(x) := \begin{bmatrix} 0 & \frac{1}{2}\frac{d}{dx_2}[s(x)b(x)] & 0 \\ \frac{1}{2}\frac{d}{dx_2}[s(x)b(x)] & 0 & \frac{1}{2}s(x)b(x) \\ 0 & \frac{1}{2}s(x)b(x) & 0 \end{bmatrix},$$

$$Z_4(x) := \begin{bmatrix} 0 & 0 & \frac{d}{dx_2}[s(x)c(x)] \\ 0 & 0 & 0 \\ \frac{d}{dx_2}[s(x)c(x)] & 0 & 2s(x)c(x) \end{bmatrix}.$$

By combining (5.5), (5.7), (5.9) and (5.10) it follows that

$$\frac{d}{dt}[V(u(t,\cdot))] = \int_\Omega q^T(t,x)Q(x)q(t,x)\,dx, \tag{5.11}$$

where for all $x \in \Omega$

$$Q(x) := \begin{bmatrix} 2s(x)f(x) & Q_{12}(x) & Q_{13}(x) \\ Q_{12}(x) & -2s(x)a(x) & -s(x)b(x) \\ Q_{13}(x) & -s(x)b(x) & -2s(x)c(x) \end{bmatrix} \tag{5.12}$$

with

$$\begin{aligned} Q_{12}(x) :&= s(x)d(x) - \frac{d}{dx_1}[s(x)a(x)] - \frac{1}{2}\frac{d}{dx_2}[s(x)b(x)], \\ Q_{13}(x) :&= s(x)e(x) - \frac{1}{2}\frac{d}{dx_1}[s(x)b(x)] - \frac{d}{dx_2}[s(x)c(x)]. \end{aligned}$$

If $Q(x) \le 0$ for all $x \in \Omega$, then the time derivative in (5.11) is clearly non-positive for all $t > 0$. However, such a condition on Q is conservative. To decrease that conservatism, matrix valued functions Υ_i are introduced such that

$$\int_\Omega q^T(t,x)\Upsilon_i(x)q(t,x)\,dx = 0$$

and, therefore, can be added to Q without altering the integral. Υ_i are spacing functions. We parameterize Υ_i by polynomials p_i.

5.3 Spacing Functions for PDEs with Two Spatial Dimensions

The following holds for any polynomial p_1, because of the boundary conditions (5.2).

$$\int_\Omega \frac{d}{dx_1}[u(t,x)p_1(x)u(t,x)]\,dx = \int_0^1 \left(u(t,x)p_1(x)u(t,x)\Big|_{x_1=0}^{x_1=1} \right) dx_2 = 0. \tag{5.13}$$

Using the chain rule, we have

$$\begin{aligned} \frac{d}{dx_1}[u(t,x)p_1(x)u(t,x)] &= u(t,x)\frac{d}{dx_1}[p_1(x)]u(t,x) + 2p_1(x)u(t,x)u_{x_1}(t,x) \\ &= q^T(t,x)\Upsilon_1(x)q(t,x), \end{aligned} \tag{5.14}$$

where for all $x \in \Omega$

$$\Upsilon_1(x) := \begin{bmatrix} \frac{d}{dx_1}[p_1(x)] & p_1(x) & 0 \\ p_1(x) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5.15}$$

Combining (5.13) and (5.14) results in

$$\int_\Omega q(t,x)^T \Upsilon_1(x) q(t,x)\,dx = 0. \tag{5.16}$$

Likewise in (5.13), because of the boundary conditions (5.2), the following holds for any polynomial p_2.

$$\int_\Omega \frac{d}{dx_2}[u(t,x)p_2(x)u(t,x)]\,dx = 0.$$

Following steps of (5.14) for $\frac{d}{dx_2}[u(t,x)p_2(x)u(t,x)]$, gives

$$\Upsilon_2(x) := \begin{bmatrix} \frac{d}{dx_2}[p_2(x)] & 0 & p_2(x) \\ 0 & 0 & 0 \\ p_2(x) & 0 & 0 \end{bmatrix} \tag{5.17}$$

such that

$$\int_\Omega q(t,x)^T \Upsilon_2(x) q(t,x)\, dx = 0. \tag{5.18}$$

Similarly to (5.13), the following is true for any polynomial p_3.

$$\int_\Omega \frac{d}{dx_2}[u(t,x)p_3(x)u_{x_1}(t,x)]\, dx = 0. \tag{5.19}$$

Note that the left-hand side of (5.19) can be written as follows.

$$\begin{aligned}\int_\Omega \frac{d}{dx_2}\left[u(t,x)p_3(x)u_{x_1}(t,x)\right]\, dx \\ = \int_\Omega \Big(u_{x_2}(t,x)p_3(x)u_{x_1}(t,x) + u(t,x)\frac{d}{dx_2}[p_3(x)]u_{x_1}(t,x)\Big)dx \\ + \int_\Omega u(t,x)p_3(x)u_{x_2x_1}(t,x)\, dx,\end{aligned} \tag{5.20}$$

where we need property (5.6). Applying integration by parts to the second integral of the right-hand side of the last equation in (5.20) results in

$$\begin{aligned}\int_\Omega u(t,x)p_3(x)\frac{d}{dx_1}\left[u_{x_2}(t,x)\right]\, dx \\ = \int_0^1 \Big(u(t,x)p_3(x)u_{x_2}(t,x)\Big|_{x_1=0}^{x_1=1} - \int_0^1 u_{x_2}(t,x)\frac{d}{dx_1}\left[u(t,x)p_3(x)\right]\, dx_1\Big)\, dx_2 \\ = -\int_\Omega u_{x_2}(t,x)\Big(u_{x_1}(t,x)p_3(x) + u(t,x)\frac{d}{dx_1}[p_3(x)]\Big)\, dx.\end{aligned} \tag{5.21}$$

From (5.20) and (5.21) the following holds.

$$\begin{aligned}\int_\Omega \frac{d}{dx_2}[u(t,x)p_3(x)u_{x_1}(t,x)]\, dx \\ = \int_\Omega \Big(u(t,x)\frac{d}{dx_2}[p_3(x)]u_{x_1}(t,x) - u(t,x)\frac{d}{dx_1}[p_3(x)]u_{x_2}(t,x)\Big)dx \\ = \int_\Omega q(t,x)^T\Upsilon_3(x)q(t,x)\, dx,\end{aligned} \tag{5.22}$$

where for all $x \in \Omega$

$$\Upsilon_3(x) := \begin{bmatrix} 0 & \frac{1}{2}\frac{d}{dx_2}[p_3(x)] & -\frac{1}{2}\frac{d}{dx_1}[p_3(x)] \\ \frac{1}{2}\frac{d}{dx_2}[p_3(x)] & 0 & 0 \\ -\frac{1}{2}\frac{d}{dx_1}[p_3(x)] & 0 & 0 \end{bmatrix}. \tag{5.23}$$

Combining (5.19) and (5.22) gives

$$\int_{\Omega} q(t,x)^T \Upsilon_3(x) q(t,x)\, dx = 0. \tag{5.24}$$

Following steps (5.19) – (5.22) for

$$\frac{d}{dx_1}[u(t,x)p_4(x)u_{x_2}(t,x)]$$

with any polynomial p_4, leads to the following.

$$\int_{\Omega} q(t,x)^T \Upsilon_4(x) q(t,x)\, dx = 0, \tag{5.25}$$

where for all $x \in \Omega$

$$\Upsilon_4(x) := \begin{bmatrix} 0 & -\frac{1}{2}\frac{d}{dx_2}[p_4(x)] & \frac{1}{2}\frac{d}{dx_1}[p_4(x)] \\ -\frac{1}{2}\frac{d}{dx_2}[p_4(x)] & 0 & 0 \\ \frac{1}{2}\frac{d}{dx_1}[p_4(x)] & 0 & 0 \end{bmatrix}. \tag{5.26}$$

5.4 LMIs for PDEs with Two Spatial Dimensions

From (5.11), (5.16), (5.18), (5.24) and (5.25) the following holds.

$$\frac{d}{dt}[V(u(t,\cdot))] = \int_{\Omega} q^T(t,x) \left(Q(x) + \sum_{i=1}^{4} \Upsilon_i(x) \right) q(t,x) dx. \tag{5.27}$$

By substituting for Q and Υ_i from (5.12), (5.15), (5.17), (5.23) and (5.26) in (5.27) one can define

$$M = \Phi(a,b,c,d,e,f,s,p_1,p_2,p_3,p_4), \tag{5.28}$$

if for all $x \in \Omega$

$$M(x) = \begin{bmatrix} M_1(x) & M_2(x) & M_3(x) \\ M_2(x) & -2s(x)a(x) & -s(x)b(x) \\ M_3(x) & -s(x)b(x) & -2s(x)c(x) \end{bmatrix}, \tag{5.29}$$

where

$$
\begin{aligned}
M_1(x) &:= 2s(x)f(x) + \frac{d}{dx_1}[p_1(x)] + \frac{d}{dx_2}[p_2(x)],\\
M_2(x) &:= s(x)d(x) - \frac{d}{dx_1}[s(x)a(x)] - \frac{1}{2}\frac{d}{dx_2}[s(x)b(x)] + p_1(x) + \frac{1}{2}\frac{d}{dx_2}[p_3(x) - p_4(x)],\\
M_3(x) &:= s(x)e(x) - \frac{1}{2}\frac{d}{dx_1}[s(x)b(x)] - \frac{d}{dx_2}[s(x)c(x)] + p_2(x) + \frac{1}{2}\frac{d}{dx_1}[p_4(x) - p_3(x)],
\end{aligned}
\tag{5.30}
$$

such that

$$
\frac{d}{dt}[V(u(t,\cdot))] = \int_\Omega q^T(t,x)M(x)q(t,x)dx. \tag{5.31}
$$

Theorem 8. *Suppose that for (5.1) there exist polynomials s, p_1, p_2, p_3, p_4 and $\theta > 0$, such that $s(x) \geq \theta$ and $M(x) \leq 0$ for all $x \in \Omega$, where M is defined as in (5.28) – (5.30). Then (5.1) is stable.*

Proof. If conditions of Theorem 8 are satisfied, let

$$
a = \inf_{x\in\Omega}\{s(x)\},\ \ b = \sup_{x\in\Omega}\{s(x)\}. \tag{5.32}
$$

Since $s(x) \geq \theta > 0$ for all $x \in \Omega$, then $b, a > 0$ and the following holds for all $v \in L_2(\Omega)$.

$$
a\|v\|_{L_2}^2 = \inf_{x\in\Omega}\{s(x)\}\int_\Omega v^2(x)\,dx \leq \int_\Omega s(x)v^2(x)\,dx \leq \sup_{x\in\Omega}\{s(x)\}\int_\Omega v^2(x)\,dx = b\|v\|_{L_2}^2. \tag{5.33}
$$

Using (5.3) it follows that

$$
a\|v\|_{L_2}^2 \leq V(v) \leq b\|v\|_{L_2}^2.
$$

Since $M(x) \leq 0$, from (5.31) it follows that $\frac{d}{dt}[V(u(t,\cdot))] \leq 0$ for all $t > 0$. Theorem 2 ensures stability of (5.1). $\square$

Theorem 9. *Suppose that for (5.1) there exist* $\theta, \gamma > 0$ *and polynomials* s, p_1, p_2, p_3, p_4 *such that* $s(x) \geq \theta$ *and* $M(x) + \gamma S(x) \leq 0$ *for all* $x \in \Omega$, *where* M *is defined as in (5.28) – (5.30) and*

$$S(x) := \begin{bmatrix} s(x) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5.34}$$

Then for all $t > 0$ *solution to (5.1) satisfies*

$$\|u(t, \cdot)\|_{L_2} \leq \sqrt{\frac{b}{a}} \|u(0, \cdot)\|_{L_2} \exp\{-\frac{\gamma}{2} t\}, \tag{5.35}$$

where a, b *are defined as in (5.32).*

Proof. Under the assumptions of Theorem 9, (5.33) holds. With (5.34) and (5.8) we can write

$$V(u(t, \cdot)) = \int_\Omega q(t, x)^T S(x) q(t, x)\, dx. \tag{5.36}$$

Since $M(x) + \gamma S(x) \leq 0$ for all $x \in \Omega$, it holds that

$$\int_\Omega q^T(t, x)(M(x) + \gamma S(x)) q(t, x)\, dx \leq 0. \tag{5.37}$$

Since γ is a scalar, (5.37) can be easily satisfied as follows.

$$\int_\Omega q^T(t, x) M(x) q(t, x) dx \leq -\gamma \int_\Omega q(t, x)^T S(x) q(t, x) dx,$$

which with (5.31) and (5.36) provides $\frac{d}{dt}[V(u(t, \cdot))] \leq -\gamma V(u(t, \cdot))$. Using proof of Theorem 2 with $c/b = \gamma$, results in (5.35). $\square$

Theorem 10. *Suppose that for (5.1) there exist polynomials* s, p_1, p_2, p_3, p_4, $\theta > 0$, *SOS polynomials* n_1, n_2, Q_1, Q_2, Q_3 *such that for all* $x_1, x_2 \in (0, 1)$

$$\begin{aligned} s(x) &= \theta + x_1(1 - x_1) n_1(x) + x_2(1 - x_2) n_2(x), \\ M(x) &= - Q_1(x) - x_1(1 - x_1) Q_2(x) - x_2(1 - x_2) Q_3(x), \end{aligned} \tag{5.38}$$

where M *is defined as in (5.28) – (5.30). Then (5.1) is stable.*

Proof. If (5.38) holds, then clearly $s(x) \geq \theta$ and $M(x) \leq 0$ for all $x \in \Omega$. Using Theorem 8 provides stability of (5.1). $\square$

Theorem 11. *Suppose that for (5.1) there exist* $\theta, \gamma > 0$, *polynomials* s, p_1, p_2, p_3, p_4, *SOS polynomials* n_1, n_2, Q_1, Q_2, Q_3 *such that for all* $x_1, x_2 \in (0,1)$

$$\begin{aligned} s(x) =& \theta + x_1(1-x_1)n_1(x) + x_2(1-x_2)n_2(x), \\ M(x) + \gamma S(x) =& -Q_4(x) - x_1(1-x_1)Q_5(x) - x_2(1-x_2)Q_6(x), \end{aligned} \tag{5.39}$$

where M is defined as in (5.28) – (5.30) and S as in (5.34), then for all $t > 0$ solution to (5.1) satisfies

$$\|u(t,\cdot)\|_{L_2} \leq \sqrt{\frac{b}{a}}\|u(0,\cdot)\|_{L_2} \exp\{-\frac{\gamma}{2}t\}, \tag{5.40}$$

where a, b are defined as in (5.32).

Proof. If (5.39) is true, then for all $x \in \Omega$, $s(x) \geq \theta$ and $M(x) + \gamma S(x) \leq 0$, which, combined with Theorem 9, gives (5.40). $\square$

Chapter 6

NUMERICAL VALIDATION

In this chapter some examples are presented to verify proposed algorithms.

6.1 Coupled PDEs

First, consider some examples of PDEs for vector valued functions.

6.1.1 Decoupled PDEs

Start with the following parameterized decoupled PDE.

$$u_t(t,x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u_{xx}(t,x) + \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} u(t,x) \tag{6.1}$$

with boundary conditions

$$u(t,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } u(t,1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The numerical solution given by MATLAB PDEPE solver implies that for $\lambda = 9.8$ (6.1) is stable and for $\lambda = 9.9$, (6.1) is unstable. Using a bisection search over λ, one may determine a lower bound on λ_{cr} for which problem in Theorem 7, with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = 0, \quad C = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \quad D = \begin{bmatrix} I_2 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.2}$$

may be shown to be feasible. Some dependence of the λ_{cr} on the degree of parameterization d is presented in Table 6.1 and compared to the λ_{cr} calculated using MATLAB PDEPE solver.

Table 6.1: Maximum λ for which (6.1) is stable based on the proposed algorithm for different degree d with $\epsilon = 0.001$.

d	1	2	3	4	5	6	λ_{num}
λ	5	5.8	7.4	8.1	8.1	8.1	9.8

6.1.2 Coupled PDEs

The following example includes some coupling.

$$u_t(t,x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u_{xx}(t,x) + \begin{bmatrix} \lambda & 1 \\ 1 & \lambda \end{bmatrix} u(t,x) \tag{6.3}$$

and boundary conditions are

$$u(t,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } u(t,1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The numerical solution given by MATLAB PDEPE solver yields that for $\lambda = 8.8$ (6.3) is stable and for $\lambda = 8.9$, (6.3) is unstable.

Using a bisection search over λ, one may determine a lower bound on λ_{cr} for which problem in Theorem 7, with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = 0, \quad C = \begin{bmatrix} \lambda & 1 \\ 1 & \lambda \end{bmatrix}, \quad D = \begin{bmatrix} I_2 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.4}$$

may be shown to be feasible. Some dependence of the λ_{cr} on the degree of parameterization d is presented in Table 6.2 and compared to the λ_{cr} calculated using MATLAB PDEPE solver.

Table 6.2: Maximum λ for which (6.3) is stable based on the proposed algorithm for different degree d with $\epsilon = 0.001$.

d	1	2	3	4	5	6	λ_{num}
λ	4	5.8	6.9	7.2	7.4	7.4	8.8

6.1.3 Coupled PDEs with Mixed Boundary Conditions

The next example includes coupled PDEs with mixed boundary conditions.

$$u_t(t,x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u_{xx}(t,x) + \begin{bmatrix} \lambda & \lambda \\ \lambda & \lambda \end{bmatrix} u(t,x) \tag{6.5}$$

where boundary conditions are

$$u_x(t,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } u(t,1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The numerical solution from MATLAB PDEPE solver states that for $\lambda = 15.9$ (6.5) is stable and for $\lambda = 16$, (6.5) is unstable.

Using a bisection search over λ, one may determine a lower bound on λ_{cr} for which problem in Theorem 7, with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = 0, \quad C = \begin{bmatrix} \lambda & \lambda \\ \lambda & \lambda \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 \\ 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.6}$$

may be shown to be feasible. Some dependence of the λ_{cr} on the degree of parameterization d is presented in Table 6.3 and compared to the λ_{cr} calculated using MATLAB PDEPE solver.

Table 6.3: Maximum λ for which (6.5) is stable based on the proposed algorithm for different degree d with $\epsilon = 0.001$.

d	1	2	3	4	5	6	λ_{num}
λ	8.6	12.7	13.9	14.4	14.6	14.7	15.9

6.1.4 Coupled PDEs with Spatially Dependent Coefficients

For our final example, we consider a coupled PDE with spatially varying coefficients.

$$u_t(t,x) = \begin{bmatrix} 5x^2+4 & 0 \\ 2x^2+7x & 7x^2+6 \end{bmatrix} u_{xx}(t,x) + \begin{bmatrix} 1 & -4x \\ -3.5x^2 & 0 \end{bmatrix} u_x(t,x) - \begin{bmatrix} x^2 & 3 \\ 2x & 3x^2 \end{bmatrix} u(t,x) \tag{6.7}$$

for all $t > 0$, $x \in (0,1)$. Also for all $t > 0$,

$$u(t,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } u(t,1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Based on the numerical solution given by MATLAB PDEPE one can say that (6.7) is stable. With $d = 4$ problem from Theorem 7 with

$$A = \begin{bmatrix} 5x^2+4 & 0 \\ 2x^2+7x & 7x^2+6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -4x \\ -3.5x^2 & 0 \end{bmatrix},$$
$$C = \begin{bmatrix} x^2 & 3 \\ 2x & 3x^2 \end{bmatrix}, \quad D = \begin{bmatrix} I_2 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.8}$$

is feasible.

6.2 Examples of PDEs with Two Spatial Variables

6.2.1 Model of Population Dynamics

In this section stability analysis is presented for the biological "KISS" PDE named after Kierstead, Slobodkin and Skelam, which describes population growth on a finite area. For more details see Holmes *et al.* (1994). The system is modeled by the following PDE.

$$u_t(t,x) = h\Big(u_{x_1x_1}(t,x) + u_{x_2x_2}(t,x)\Big) + ru(t,x), \tag{6.9}$$

where $h, r > 0$, $x \in \Omega \subset \mathbb{R}^2$ and scalar function u satisfies homogeneous Dirichlet boundary conditions.

It is claimed in Holmes *et al.* (1994) that if Ω is a square with edge of length l, then

$$l_{cr} := \sqrt{2\pi^2(\frac{h}{r})} \tag{6.10}$$

defines a critical length. That means, if $l > l_{cr}$, then (6.9) is unstable. Alternatively, for given l and r (6.10) defines h_{cr} as

$$h_{cr} := l^2 r/2\pi^2. \tag{6.11}$$

Therefore, if $h < h_{cr}$, then (6.9) is unstable.

For testing the proposed algorithm, fix $l = 1$ and arbitrarily choose r, for example $r = 4$. Thus, according to (6.11), $h_{cr} \approx 0.203$.

Using a bisection search over h, one can determine a lower bound of h_{cr} for which SOS problem in Theorem 10, with

$$a(x) = h, \quad b(x) = 0, \quad c(x) = h, \quad d(x) = 0, \quad e(x) = 0, \quad f(x) = 4 \quad \text{for all } x \in \Omega \tag{6.12}$$

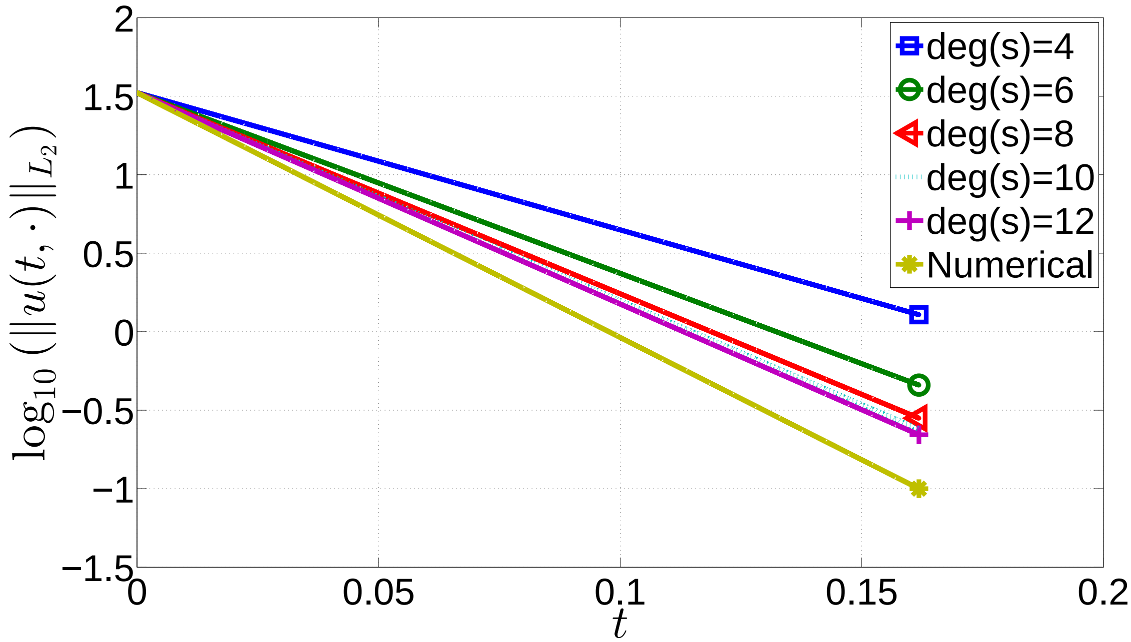

Figure 6.1: Semi-log plots of the L_2 norm of the numerical solution to (6.9) with $u(0,x) = 10^3 x_1 x_2 (1 - x_1)(1 - x_2)$ and bounds, given by the proposed method with different $\deg(s)$.

may be shown to be feasible. Results for different degrees of s (deg(s)) are presented in Table (6.4).

Now we choose $l = 1$, $h = 2$ and $r = 4$. Using a bisection search over γ, we determine the maximum γ for which the SOS problem in Theorem 11, with (6.12), may be shown to be feasible. Results for different $\deg(s)$ are presented in Table (6.5). Using finite difference scheme, we numerically solve (6.9) with $u(0,x) = 10^3 x_1 x_2 (1 - x_1)(1-x_2)$. Plots of $\log_{10}(\|u(t,\cdot)\|_{L_2})$ versus t, using a numerical solution, and bounds on $\log_{10}(\|u(t,\cdot)\|_{L_2})$, given by the proposed method for different $\deg(s)$, are presented in Fig. (6.1). These plots allow us to determine γ by examining the rate of decrease in the L_2 norm. Plots are aligned at $t = 0$ in order to better compare our SOS estimates of γ to the estimate of γ derived from numerical simulation as a function of increasing $\deg(s)$.

Table 6.4: Minimum h_{cr} vs deg(s) for (6.9)

deg(s)	4	6	8	10	12	analytic
h_{cr}	0.332	0.259	0.238	0.229	0.227	0.203

Table 6.5: Maximum γ vs deg(s) for (6.9) with $h = 2$

deg(s)	4	6	8	10	12
γ	40.25	53	59	61	62

6.2.2 Random PDE with Spatially Dependent Coefficients

Consider

$$\begin{aligned} u_t(t,x) &= (5x_1^2 - 15x_1x_2 + 13x_2^2)(u_{x_1x_1}(t,x) + u_{x_2x_2}(t,x)) + (10x_1 - 15x_2)u_{x_1}(t,x) \\ &\quad + (-15x_1 + 26x_2)u_{x_2}(t,x) - (17x_1^4 - 30x_2 - 25x_1^2 - 8x_2^3 - 50x_2^4)u(t,x), \\ u(0,x) &= 10^3 x_1x_2(1-x_1)(1-x_2) \end{aligned} \tag{6.13}$$

where $x \in \Omega := (0,1)^2$ and the scalar function u satisfies zero Dirichlet boundary conditions.

Using a bisection search over γ, we determine maximum γ for which SOS problem in Theorem 11, with

$$a(x) = 5x_1^2 - 15x_1x_2 + 13x_2^2, \quad e(x) = -15x_1 + 26x_2, \quad c(x) = 5x_1^2 - 15x_1x_2 + 13x_2^2,$$
$$d(x) = 10x_1 - 15x_2, \quad f(x) = -(17x_1^4 - 30x_2 - 25x_1^2 - 8x_2^3 - 50x_2^4), \quad b(x) = 0$$

may be shown to be feasible.

Using finite difference scheme, we numerically solve (6.13). The estimated rate of decay, based on numerical solution, is 13.07. The computed rate of decay, based on our SOS method, is 12.5 for deg(s) = 8. Plots are given in Fig. 6.2 and, as for

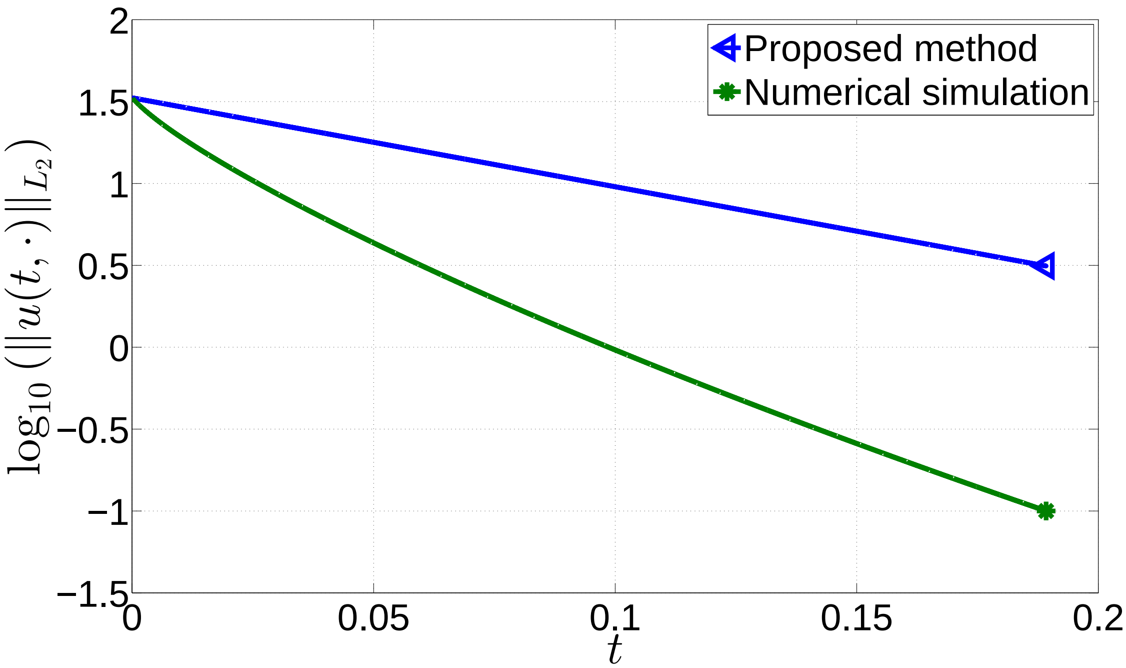

Figure 6.2: Semi-log plots of the L_2 norm of the numerical solution to (6.13) and bound, given by the proposed method with $\deg(s) = 8$.

Fig. 6.1, are aligned at $t = 0$.

Chapter 7

CONCLUSION

7.1 Summary of Contribution

In this thesis, we have presented a computational framework based on convex optimization for stability analysis of two forms of linear PDEs. First form includes coupled PDEs with spatially varying polynomial coefficients. Second form considers parabolic PDEs for scalar-valued functions with two spatial variables.

We used LMIs and SOS to parameterize positive functionals. We have enforced negativity of the derivative using a combination of SOS and a parametrization of projection operators defined by the fundamental theorem of calculus. The result is an LMI test for stability which can be implemented using SOSTOOLS coupled with an SDP solver such as Mosek or SeDuMi. We applied the proposed framework to several examples of systems of coupled linear PDEs with both constant and spatially varying coefficients and with both Dirichlet and Neumann boundary conditions. Also we calculated an upper bound on the rate of decay of the L_2 norm of a solution to PDE which describes dynamics of population. We compared the numerical results with solutions based on discretization methods.

7.2 Ongoing Research

Future work includes extension of the framework to study stability of models such as the acoustic wave equations as well as examine the problem of optimal control and estimation for systems of coupled PDEs. Another step is the combination of presented techniques in order to study stability of coupled PDEs with multiple spatial variables.

And finally, decrease the conservatism by using semi-separable kernels to parameterize integral operators as in Gahlawat and Peet (2015).

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